1. Dk, že I. $\nabla\cdot(\nabla\times\vec{A})=0$

II. $\nabla\times(\nabla\varphi)=0$.

∇ „=" $\left(\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_3}\right)$, důkaz po složkách:

I. $$\nabla\times A = \left(\frac{\partial A_3}{\partial\xi_2}-\frac{\partial A_2}{\partial\xi_3},\ \frac{\partial A_1}{\partial\xi_3}-\frac{\partial A_3}{\partial\xi_1},\ \frac{\partial A_2}{\partial\xi_1}-\frac{\partial A_1}{\partial\xi_2}\right)$$

$$\nabla\cdot(\nabla\times A) = \frac{\partial^2 A_3}{\partial\xi_2\partial\xi_1}-\frac{\partial^2 A_2}{\partial\xi_1\partial\xi_3}+\frac{\partial^2 A_1}{\partial\xi_3\partial\xi_2}-\frac{\partial^2 A_3}{\partial\xi_1\partial\xi_2}+\frac{\partial^2 A_2}{\partial\xi_1\partial\xi_3}-\frac{\partial^2 A_1}{\partial\xi_3\partial\xi_2}=0.$$ ■

II. $$\nabla\varphi = \left(\frac{\partial\varphi}{\partial\xi_1}, \frac{\partial\varphi}{\partial\xi_2}, \frac{\partial\varphi}{\partial\xi_3}\right)$$

$$\nabla\times(\nabla\varphi) = \left(\frac{\partial^2\varphi}{\partial\xi_3\partial\xi_2}-\frac{\partial^2\varphi}{\partial\xi_2\partial\xi_3},\ \frac{\partial^2\varphi}{\partial\xi_1\partial\xi_3}-\frac{\partial^2\varphi}{\partial\xi_3\partial\xi_1},\ \frac{\partial^2\varphi}{\partial\xi_2\partial\xi_1}-\frac{\partial^2\varphi}{\partial\xi_1\partial\xi_2}\right)=\vec{0}.$$ ■

2. Využiju dvě identity: I. vektorový Laplace $\nabla^2\vec{A}=\nabla(\nabla\cdot\vec{A})-\nabla\times(\nabla\times\vec{A})$

II. divergence vekt. součinu $\nabla\cdot(\vec{\alpha}\times\vec{\beta})=(\nabla\times\vec{\alpha})\cdot\vec{\beta}-(\nabla\times\vec{\beta})\cdot\vec{\alpha}$

$$\nabla\cdot\left(\vec{A}\times(\nabla\times\vec{A})\right)\overset{\text{II.}}{=}(\nabla\times\vec{A})\cdot(\nabla\times\vec{A})-\vec{A}\cdot\left(\nabla\times(\nabla\times\vec{A})\right)\overset{\text{I.}}{=}$$

$$\overset{\text{I.}}{=}(\nabla\times\vec{A})\cdot(\nabla\times\vec{A})-\vec{A}\cdot\left(-\nabla^2\vec{A}+\nabla(\nabla\cdot\vec{A})\right)$$

$$\Rightarrow (\nabla\times\vec{A})\cdot(\nabla\times\vec{A})=\nabla\cdot\left(\vec{A}\times(\nabla\times\vec{A})\right)-\vec{A}\cdot\nabla^2\vec{A}+\vec{A}\cdot\nabla(\nabla\cdot\vec{A})$$

$$\int d^3\vec{r}\,(\nabla\times\vec{A})\cdot(\nabla\times\vec{A})=\underbrace{\int d^3\vec{r}\,\nabla\cdot\left(\vec{A}\times(\nabla\times\vec{A})\right)}_{\text{Gaussova věta} \Rightarrow \vec{A}\cdot(\nabla\times\vec{A}) \text{ musí být spojitě diferencovatelné}}-\int d^3\vec{r}\,\vec{A}\cdot\nabla^2\vec{A}+\int d^3\vec{r}\,\vec{A}\cdot\nabla\underbrace{(\nabla\cdot\vec{A})}_{\text{id. }0}=$$

$$=\underbrace{\lim_{r\to\infty}\int_{\Omega(r)} d^2\vec{r}\,\vec{A}\times(\nabla\times\vec{A})}-\int d^3\vec{r}\,\vec{A}\,\nabla^2\vec{A}$$

$\lim\limits_{r\to\infty}\left|\int_{\Omega(r)} d^2\vec{r}\,\vec{A}\times(\nabla\times\vec{A})\right| < \lim\limits_{r\to\infty}\int d^2\vec{r}\,|\vec{A}|\,|\nabla\times\vec{A}|$, tzn. pokud aspoň $|\vec{A}|\sim\frac{1}{r}$, a tak $|\nabla\times\vec{A}|\sim\frac{1}{r^2}$, pak $\lim\limits_{r\to\infty}\int d^2\vec{r}\,|\vec{A}|\,|\nabla\times\vec{A}|=0$. Pak platí

$$\underline{\int d^3\vec{r}\,(\nabla\times\vec{A})\cdot(\nabla\times\vec{A})=-\int d^3\vec{r}\,\vec{A}\cdot\nabla^2\vec{A}}$$ ■

3. Dk, že $\phi(r,t) = f(\vec{n}\cdot\vec{r}-ct)$, $|\vec{m}|=1$, řeší $\Delta\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = 0$,

$$\Delta\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

$$\underline{L} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = \frac{1}{c^2}\frac{\partial}{\partial t} f'\cdot(-c) = \frac{1}{c^2}\, f''(-c)^2 = \underline{f''}$$

$$\underline{P} = \Delta\phi = \frac{\partial^2\phi}{\partial \xi_1^2} + \frac{\partial^2\phi}{\partial \xi_2^2} + \frac{\partial\phi}{\partial \xi_3^3} = \left(\frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} + \frac{\partial^2}{\partial \xi_3^2}\right) f(\vec{m}\cdot(\vec{i}\,\xi_1 + \vec{k}\,\xi_2 + \vec{e}\,\xi_3) - ct) =$$

$$= m_x^2 f'' + m_y^2 f'' + m_z^2 f'' = m_x^2 |\vec{m}|^2 f'' = \underline{f''}, \text{ kde } f'' = \frac{\partial^2 f}{\partial \chi^2} \quad \chi = \vec{m}\cdot\vec{r} - ct$$

$$L = P \qquad \blacksquare$$