Seminární úloha 2.3 Martin Hanák

a) $D_k = \sum_{k=0}^{N} (k - Np)^2 \cdot \binom{N}{k} p^k (1-p)^{N-k} =$

$= \sum_{k=0}^{N} (k^2 - 2kNp + N^2p^2) \binom{N}{k} p^k (1-p)^{N-k}$

$= \langle k^2 \rangle - 2Np \langle k \rangle + N^2p^2 = \langle k^2 \rangle - N^2p^2$

$\langle k^2 \rangle = \sum_{k=0}^{N} k^2 \binom{N}{k} p^k (1-p)^{N-k} = \sum_{k=1}^{N} k \cdot \frac{N!}{(N-k)!\,(k-1)!} p^k (1-p)^{N-k} = \Big|\, \begin{matrix} l = k-1 \\ k = l+1 \end{matrix}$

$= \sum_{l=0}^{N-1} (l+1) \frac{N!}{(N-l-1)!\; l!} p^{l+1} \cdot (1-p)^{N-l-1} = \Big|\, \begin{matrix} M = N-1 \\ N = M+1 \end{matrix}$

$= \sum_{l=0}^{M} (l+1) \frac{(M+1)!}{(M-l)!\; l!} p^{l+1} \cdot (1-p)^{M-l} =$

$= (M+1) p \sum_{l=0}^{M} (l+1) \binom{M}{l} p^l (1-p)^{M-l} =$

$= (M+1) p \cdot (\langle l \rangle + 1) = (M+1) p \cdot (Mp + 1) =$

$= Np \cdot ((N-1)p + 1) = Np(Np - p + 1)$

$= N^2p^2 - Np^2 + Np$

$D_k = \langle k^2 \rangle - N^2p^2 = -Np^2 + Np$

$\underline{\underline{D_k = Np \cdot (1-p)}}$

b) $\mu_k^{3c} = \sum_{k=0}^{N} (k - Np)^3 \binom{N}{k} p^k (1-p)^{N-k} =$

$= \sum_{k=0}^{N} (k^3 - 3k^2Np + 3kN^2p^2 - N^3p^3) \binom{N}{k} p^k (1-p)^{N-k} =$

$= \langle k^3 \rangle - 3Np \langle k^2 \rangle + 3N^2p^2 \langle k \rangle - N^3p^3$

- platí: $\langle k \rangle = Np$

$\langle k^2 \rangle = D_k + \langle k \rangle^2 = Np(1-p) + N^2p^2 = Np - Np^2 + N^2p^2$

$\langle k^3 \rangle = ?$

$$\langle k^3\rangle = \sum_{k=0}^{N} k^3 \binom{N}{k} p^k (1-p)^{N-k} = \sum_{k=1}^{N} k^2 \frac{N!}{(N-k)!\,(k-1)!}\, p^k (1-p)^{N-k} = \left|\; \begin{matrix} l = k-1 \\ k = l+1 \end{matrix}\right.$$

$$= \sum_{l=0}^{N-1} (l+1)^2 \frac{N!}{(N-l-1)!\, l!} \cdot p^{l+1} (1-p)^{N-l-1} = \left|\; \begin{matrix} M = N-1 \\ N = M+1 \end{matrix}\right.$$

$$= \sum_{l=0}^{M} (l^2+2l+1) \frac{(M+1)!}{(M-l)!\, l!}\, p^{l+1} (1-p)^{M-l} =$$

$$= (M+1)\cdot p \cdot \sum_{l=0}^{M} (l^2+2l+1) \binom{M}{l} p^l (1-p)^{M-l} =$$

$$= (M+1)p \cdot (\langle l^2\rangle + 2\langle l\rangle + 1) = (M+1)p\cdot(Mp(1-p) + M^2p^2 + 2Mp + 1) =$$

$$= Np \cdot ((N-1)(p-p^2) + (N-1)^2p^2 + 2p\cdot(N-1) + 1) =$$

$$= Np \cdot (Np - Np^2 - p + p^2 + N^2p^2 - 2Np^2 + p^2 + 2Np - 2p + 1)$$

$$= Np \cdot (N^2p^2 - 3Np^2 + 3Np + 2p^2 - 3p + 1)$$

$$\mu_k^{3c} = N^3p^3 - 3N^2p^3 + 3N^2p^2 + 2Np^3 - 3Np^2 + Np$$
$$- 3Np\cdot(Np - Np^2 + N^2p^2) + 3N^3p^3 - N^3p^3 =$$

$$= 2Np^3 - 3Np^2 + Np = Np\,(2p^2 - 3p + 1) = Np \cdot 2\cdot(p-1)\left(p-\frac{1}{2}\right) =$$

$$\underline{\underline{\mu_k^{3c} = Np\,.\,(1-p)(1-2p)}}$$

Seminární úloha 2.4.

a) $$\langle k\rangle = \sum_{k=0}^{\infty} k \cdot \frac{\mu^k}{k!} e^{-\mu} = \sum_{k=1}^{\infty} k \cdot \frac{\mu^k}{k!} e^{-\mu} = \left|\; \begin{matrix} l = k-1 \\ k = l+1 \end{matrix}\right.$$

$$= \sum_{l=0}^{\infty} \frac{\mu^{l+1}}{l!} e^{-\mu} = \mu \cdot \sum_{l=0}^{\infty} \frac{\mu^l}{l!} e^{-\mu} = \mu$$

$$\underline{\underline{\langle k\rangle = \mu}}$$

b) $$D_k = \sum_{k=0}^{\infty} (k-\mu)^2 \frac{\mu^k}{k!} e^{-\mu} = \sum_{k=0}^{\infty} (k^2 - 2k\mu + \mu^2) \frac{\mu^k}{k!} e^{-\mu} =$$

$$= \langle k^2\rangle - 2\mu^2 + \mu^2 = \langle k^2\rangle - \mu^2$$

$$\langle k^2\rangle = \sum_{k=0}^{\infty} k^2 \frac{\mu^k}{k!} e^{-\mu} = \sum_{k=1}^{\infty} k \cdot \frac{\mu^k}{(k-1)!} e^{-\mu} = \left|\; \begin{matrix} l = k-1 \\ k = l+1 \end{matrix}\right.$$

$$= \sum_{l=0}^{\infty} (l+1) \frac{\mu^{l+1}}{l!} e^{-\mu} = \mu\cdot(\langle l\rangle + 1) = \mu(\mu+1) = \mu^2 + \mu$$

$$D_k = \mu^2 + \mu - \mu^2$$

$$\underline{D_k = \mu}$$

Seminární úloha 2.4. Martin Hanák

c) $\mu_k^{3c} = \sum_{k=0}^{\infty} (k-\mu)^3 \frac{\mu^k}{k!} e^{-\mu} = \sum_{k=0}^{\infty} (k^3 - 3k^2\mu + 3k\mu^2 - \mu^3) \frac{\mu^k}{k!} e^{-\mu} =$

$= \langle k^3 \rangle - 3\mu \langle k^2 \rangle + 3\mu^2 \langle k \rangle - \mu^3$

- platí: $\langle k \rangle = \mu$

$\langle k^2 \rangle = D_k + \mu^2 = \mu + \mu^2$

$\langle k^3 \rangle = ?$

$\langle k^3 \rangle = \sum_{k=0}^{\infty} k^3 \frac{\mu^k}{k!} e^{-\mu} = \sum_{k=1}^{\infty} k^2 \cdot \frac{\mu^k}{(k-1)!} e^{-\mu} = \left| \begin{array}{l} l = k-1 \\ k = l+1 \end{array} \right.$

$= \sum_{l=0}^{\infty} (l+1)^2 \frac{\mu^{l+1}}{l!} e^{-\mu} = \mu \cdot \left(\sum_{l=0}^{\infty} (l^2 + 2l + 1) \frac{\mu^l}{l!} e^{-\mu} \right) =$

$= \mu \cdot (\langle l^2 \rangle + 2\langle l \rangle + 1) = \mu \cdot (\mu + \mu^2 + 2\mu + 1)$

$= \mu^3 + 3\mu^2 + \mu$

$\mu_k^{3c} = \mu^3 + 3\mu^2 + \mu - 3\mu(\mu + \mu^2) + 3\mu^3 - \mu^3 =$

$= \mu^3 + 3\mu^2 + \mu - 3\mu^2 - 3\mu^3 + 3\mu^3 - \mu^3 =$

$= \mu$

$\gamma = \frac{\mu_k^{3c}}{(\mu_k^{2c})^{\frac{3}{2}}} = \frac{\mu}{\mu^{\frac{3}{2}}} = \mu^{-\frac{1}{2}}$ $\qquad \underline{\underline{\gamma = \mu^{-\frac{1}{2}}}}$

Seminární úloha 2.5.

a) $\langle x \rangle = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \left| \begin{array}{ll} t = \frac{x-\mu}{\sqrt{2}\sigma} & x = \sqrt{2}\cdot\sigma\cdot t + \mu \\ dt = \frac{1}{\sqrt{2}\sigma} dx & dx = \sqrt{2}\sigma\, dt \end{array} \right|$

$= \int_{-\infty}^{\infty} \frac{\sqrt{2}\sigma t + \mu}{\sigma\sqrt{2\pi}} e^{-t^2} \cdot \sqrt{2}\sigma\, dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu) e^{-t^2} dt =$

$= \frac{1}{\sqrt{\pi}} \cdot \left(\sqrt{2} \cdot \sigma \cdot \underbrace{\int_{-\infty}^{\infty} t \cdot e^{-t^2} dt}_{\text{lichá funkce}} + \mu \cdot \int_{-\infty}^{\infty} e^{-t^2} dt \right) =$

$= \frac{1}{\sqrt{\pi}} \cdot (0 + \mu \cdot \sqrt{\pi}) = \mu$

$\underline{\underline{\langle x \rangle = \mu}}$

b) $\cancel{D_x}^{D_x} = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \left| \begin{matrix} t = \frac{x-\mu}{\sigma} & \sigma t = x-\mu \\ dt = \frac{1}{\sigma} dx & \sigma dt = dx \end{matrix} \right|$

$= \int_{-\infty}^{\infty} \sigma^2 t^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt =$

$= \frac{\sigma^2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} t^2 e^{-\frac{t^2}{2}} dt = \sigma^2 \cdot \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = \sigma^2$

$\cancel{D_x}$ $\underline{\underline{D_x = \sigma^2}}$

c) $\mu_x^{3c} = \int_{-\infty}^{\infty} (x-\mu)^3 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \left| \begin{matrix} t = \frac{x-\mu}{\sigma} & \sigma \cdot t = x-\mu \\ dt = \frac{1}{\sigma} dx & \sigma dt = dx \end{matrix} \right|$

$= \int_{-\infty}^{\infty} \sigma^3 t^3 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt =$

$= \frac{\sigma^3}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \underbrace{t^3 e^{-\frac{t^2}{2}}}_{\text{lichá funkce}} dt =$

$= \frac{\sigma^3}{\sqrt{2\pi}} \cdot 0 = 0$

$\gamma = \frac{\mu_x^{3c}}{(\mu_x^{2c})^{\frac{3}{2}}} = \frac{0}{(\sigma^2)^{\frac{3}{2}}} = 0$

$\underline{\underline{\gamma = 0}}$