4) $\vec{B} = \nabla \times \vec{A}$ | $\nabla \cdot \vec{B} = 0$ | $\varepsilon_0 \nabla \cdot \vec{E} = \rho$

$\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \nabla\phi$ | $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ | $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$

+ Coloumbova kalibrace : $\nabla \cdot \vec{A} = 0$

a) rovnice pro ϕ:

$$\left.\begin{aligned} &\varepsilon_0 \nabla \cdot \vec{E} = \rho \\ &\varepsilon_0 \nabla \cdot \left(-\frac{\partial}{\partial t}\vec{A} - \nabla\phi\right) = \rho \\ &-\varepsilon_0 \left(\frac{\partial}{\partial t}\underbrace{\nabla \cdot \vec{A}}_{=0 \text{ z kalibrace}} + \Delta\phi\right) = \rho \end{aligned}\right\} \quad \boxed{\Delta\phi = -\frac{\rho}{\varepsilon_0}}$$

... Poissonova rovnice

b) rovnice pro $\vec{A}$

$$\nabla \times \vec{B} - \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\vec{E} = \mu_0 \vec{j}$$

$$\nabla \times \nabla \times \vec{A} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\vec{A} + \nabla\phi\right) = \mu_0 \vec{j}$$

$$-\Delta\vec{A} + \nabla(\underbrace{\nabla \cdot \vec{A}}_{=0 \text{ z kalibrace}}) + \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}\nabla\underbrace{\phi}_{\int d\vec{r}' \frac{\rho(\vec{r}', t)}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|}} = \mu_0 \vec{j}$$

$$\boxed{\Delta\vec{A} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2}\vec{A} = \mu_0 \vec{j} - \mu_0 \varepsilon_0 \nabla \int d\vec{r}' \frac{\frac{\partial}{\partial t}\rho(\vec{r}', t)}{4\pi\varepsilon_0 |\vec{r} - \vec{r}'|}}$$

---

5) $E_x(\vec{r}, t) = E_x(t) \cos(k_x x) \sin(k_y y) \sin(k_z z)$ + bez náboje

$E_y(\vec{r}, t) = E_y(t) \sin(k_x x) \cos(k_y y) \sin(k_z z)$

$E_z(\vec{r}, t) = E_z(t) \sin(k_x x) \sin(k_y y) \cos(k_z z)$

a) dokažte $\vec{k} \cdot \vec{E}(t) = 0$

Maxwellova rovnice $\varepsilon_0 \nabla \cdot \vec{E} = \rho$ → bez náboje : $\nabla \cdot \vec{E} = 0$

$$\rightarrow \nabla \cdot \vec{E}(\vec{r}, t) = -E_x(t) k_x \sin(k_x x) \sin(k_y y) \sin(k_z z) - E_y(t) k_y \sin() \sin() \sin() -$$

$$- E_z(t) k_z \sin() \sin() \sin() = \vec{E}(t) \cdot \vec{k} \; \underbrace{\sin(k_x x) \sin(k_y y) \sin(k_z z)}_{\neq 0 \text{ identicky}} = 0$$

$$\Rightarrow \boxed{\vec{E}(t) \cdot \vec{k} = 0} \; \blacksquare$$

b) $\underline{\text{určit } \vec{E}(t)}$

$$\left.\begin{array}{l} \nabla\times\vec{B} - \mu_0\varepsilon_0 \frac{\partial}{\partial t}\vec{E} = \mu_0\vec{j} \\ \nabla\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0 \end{array}\right\} \begin{array}{l} \nabla\times(\nabla\times\vec{E}) + \frac{\partial}{\partial t}\nabla\times\vec{B} = 0 \\ -\Delta\vec{E} + \nabla(\nabla\cdot\vec{E}) + \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E} + \mu_0\frac{\partial}{\partial t}\vec{j} = 0 \end{array}$$

→ předpokládám izotropní a homogenní prostředí bez proudů ⇒ dostávám vlnovou rovnici:

$$\Delta\vec{E} - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

$$(\Delta\vec{E}(\vec{r},t))_x = - E_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z)\underbrace{(k_x^2+k_y^2+k_z^2)}_{|\vec{k}|^2}$$

+ analogicky pro další složky

→ v x-složce máme:

$$-k^2E_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z) - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}E(t)\cdot\cos(k_x x)\sin(k_y y)\sin(k_z z) = 0$$

$$E''(t) + \underbrace{\frac{k^2}{\mu_0\varepsilon_0}}_{\omega^2}E(t) = 0$$ ... ODR 2. řádu → hledáme řešení jako $e^{\alpha t}$

$\alpha^2 + \omega^2 = 0 \rightarrow \alpha = \pm i\omega$ ... volím trigonometrický systém

$$\Rightarrow \left| \begin{array}{l} E_x(t) = E_x\sin(\omega t+\varphi_x) \\ E_y(t) = E_y\sin(\omega t+\varphi_y) \\ E_z(t) = E_z\sin(\omega t+\varphi_z) \end{array} \right|$$

c) $\underline{\text{určit magnetickou složku pole}}$

$E_x(t) = E_x\sin(\omega t+\varphi_x)\cos(k_x x)\sin(k_y y)\sin(k_z z)$ atd. pro další složky

$$\rightarrow (\nabla\times\vec{E})_x = -\frac{\partial\vec{B}_x}{\partial t} = E_z k_y\sin(\omega t+\varphi_z)\sin(k_x x)\cos(k_y y)\cos(k_z z) - E_y k_z\sin(\omega t+\varphi_y)\sin(k_x x)\cos(k_y y)\cos(k_z z) =$$

$$= \left[E_z k_y\sin(\omega t+\varphi_z) - E_y k_z\sin(\omega t+\varphi_y)\right]\sin(k_x x)\cos(k_y y)\cos(k_z z)$$

$$B_x = -\sin(k_x x)\cos(k_y y)\cos(k_z z)\int E_z k_y\sin(\omega t+\varphi_z) - E_y k_z\sin(\omega t+\varphi_y)\,dt$$

$$B_x = \left[E_y \frac{k_z}{\omega} \cos(\omega t + \varphi_y) - E_z \frac{k_y}{\omega} \cos(\omega t + \varphi_z) \right] \sin(k_x x) \cos(k_y y) \cos(k_z z)$$

$$B_y = \left[E_z \frac{k_x}{\omega} \cos(\omega t + \varphi_z) - E_x \frac{k_z}{\omega} \cos(\omega t + \varphi_x) \right] \cos(k_x x) \sin(k_y y) \cos(k_z z)$$

$$B_z = \left[E_x \frac{k_y}{\omega} \cos(\omega t + \varphi_x) - E_y \frac{k_x}{\omega} \cos(\omega t + \varphi_y) \right] \cos(k_x x) \cos(k_y y) \sin(k_z z)$$

... analogicky i pro ostatní složky

---

6) Dk, že :

$$\sum_{\lambda=1,2} \vec{e}^{\,\mu}_{\lambda,\vec{q}}\, \vec{e}^{\,\nu *}_{\lambda,\vec{q}} = \delta^{\mu\nu} - \frac{\vec{q}^{\,\mu} \vec{q}^{\,\nu}}{\vec{q}^{\,2}}, \text{ pokud } \vec{q} \cdot \vec{e}_{\lambda,\vec{q}} = 0.$$

$$\sum_{\lambda=1,2} \vec{e}^{\,\mu}_{\lambda,\vec{q}}\, \vec{e}^{\,\nu *}_{\lambda,\vec{q}} = \delta^{\mu\nu} - \frac{\vec{q}^{\,\mu} \vec{q}^{\,\nu}}{\vec{q}^{\,2}} \qquad / \cdot q^{\mu}$$

$$\underbrace{\sum \vec{e}^{\,\mu}_{\lambda,\vec{q}}\, q^{\mu}}_{0} \vec{e}^{\,\nu *}_{\lambda,\vec{q}} = \delta^{\mu\nu} q^{\mu} - \frac{\vec{q}^{\,\mu} \vec{q}^{\,\mu} \vec{q}^{\,\nu}}{q^2}$$

$$= q^{\nu} - \frac{\cancel{q^2}\, q^{\nu}}{\cancel{q^2}}$$

$$0 = q^{\nu} - q^{\nu} = 0$$

$$0 = 0. \quad \blacksquare$$