1. $y' = \alpha y (P_m - y) \qquad y(0) = y_0 \in (0, P_m)$

- $x \in \mathbb{R} \quad J \in \mathbb{R}$
- $y: \quad y_0 \equiv 0 \qquad y_0 = P_m \qquad P_m > 0$
- $I_1 \ (-\infty, 0) \qquad \underbrace{I_2 = (0, P_m)}_{\text{zač. podmínka}} \qquad I_3 = (P_m, \infty)$

- $$\int \frac{dy}{\alpha y (P_m - y)} = \int dx + C$$

$$\frac{1}{P_m \alpha} \int \frac{1}{y} + \frac{1}{P_m - y} \, dy = (\ln|y| - \ln|P_m - y|)\frac{1}{P_m \alpha}$$

$$= \quad P_m - y > 0 \qquad y \in (0, P_m)$$

$$\frac{1}{P_m \alpha}\left(\ln \frac{y}{P_m - y}\right) = x + C \ \rightarrow \ \frac{y}{P_m - y} = K e^{P_m \alpha x}$$

$$y(0) = y_0$$

$$\frac{y_0}{P_m - y_0} = K$$

$$\frac{y - P_m + P_m}{P_m - y} = -1 + \frac{P_m}{P_m - y} = K e^{x P_m \alpha}$$

$$\frac{P_m}{P_m - y} = 1 + K e^{x P_m \alpha}$$

$$\frac{P_m}{K e^{x P_m \alpha} + 1} - P_m = y$$

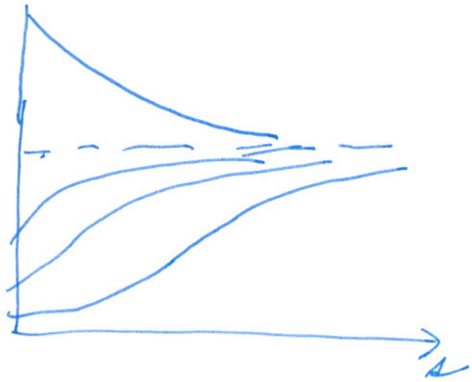
$$\frac{P_m}{\frac{y_0}{P_m - y_0} e^{x P_m \alpha} + 1} - P_m = y$$

$$\frac{P_m (P_m - y_0)}{P_m e^{x \alpha P_m} + (P_m - y_0)} - P_m = y$$

$$y' = r y \left(1 - \frac{y}{K}\right)$$

growth rate
relative growth

carrying
capacity



$$y' = \frac{r}{K} y (K - y)$$

α

$$f(x) = \frac{L}{1+e^{-k(x-x_0)}}$$ logistic function

$$P(t) = \frac{K}{1+Ae^{-rt}}$$

$$A = \frac{K - P_0}{P_0} \quad P_0 = P(t_0)$$

(2) $y' = \frac{\sqrt{y}}{\sqrt{x}}$ $\quad x \neq 0$ $\quad y'$ vždy kladná

→ rostoucí fce

- $x: I = (0, \infty)$
- $y \geq 0 \quad g(0) = 0 \rightarrow y_0 = 0$

$y: J \quad (0, \infty)$

- $\frac{y'}{\sqrt{y}} = \frac{1}{\sqrt{x}}$

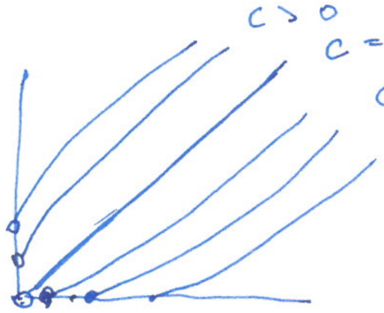
$2\sqrt{y} = 2\sqrt{x} + \tilde{c}$

$\sqrt{y} = \sqrt{x} + c \quad x \in (0, \infty)$

$\sqrt{y} \geq 0 \rightarrow \underline{\sqrt{x} + c \geq 0}$

$c \geq -\sqrt{x}$

- řešení: $y = (\sqrt{x} + c)^2$



(3) $y' = \frac{1-x}{y}$ $\quad y \neq 0$

- $x: \quad I = \mathbb{R} \quad f(x) = 1 - x$
- $y \neq 0 \quad g(y) = \frac{1}{y}$

$y: J_1 = (-\infty, 0) \quad J_2 = (0, \infty)$

- $y y' = 1 - x$

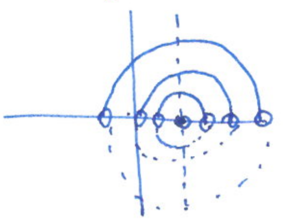
$\frac{y^2}{2} = x - \frac{x^2}{2} + \tilde{c} \quad \tilde{c} \in \mathbb{R}$

$y^2 = -x^2 + 2x + c$

řešení

$y > 0 \quad y = +\sqrt{-x^2 + 2x + c}$

$y < 0 \quad y = -\sqrt{-x^2 + 2x + c}$



POZOR

$-x^2 + 2x + c \geq 0$

y^2 vždy kladné.

? kdy má smysl

kořeny $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{-2 \pm \sqrt{4 + 4c}}{-2} = 1 \mp \sqrt{1 + c}$

pro $c < -1$ vždy záporné

$\Rightarrow c \geq -1$

$x \in [1 - \sqrt{1+c}, 1 + \sqrt{1+c}]$

(4) $y' = -\dfrac{e^x}{2y(1+e^x)}$ $\qquad y \neq 0$

· x: $I = \mathbb{R}$, $1+e^x > 0 \ \forall x$

· $y \neq 0$ žádné stacionární řešení

· $-2y\,y' = \dfrac{e^x}{1+e^x}$

$-\int 2y\,dy = \int \dfrac{e^x}{1+e^x}\,dx + C$

$-y^2 = \ln(\underbrace{1+e^x}_{>0\ \forall x}) + \tilde{C}$

$y^2 = -\ln(1+e^x) + C$

$y > 0 \quad \sqrt{C - \ln(1+e^x)}$

$y < 0 \quad -\sqrt{C - \ln(1+e^x)}$

$-\ln(1+e^x) + C \geq 0$

protože $y^2 \geq 0$

$C \geq \ln(1+e^x)$

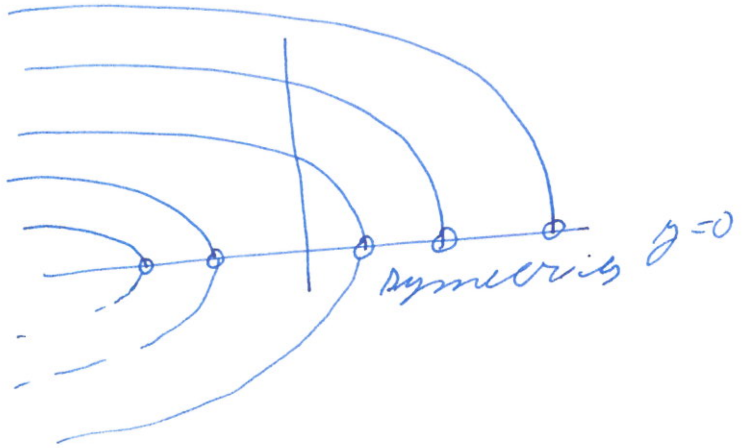
$C \geq \ln(1+e^x)$

$e^C \geq 1+e^x$

$e^C - 1 \geq e^x$

$\ln(e^C - 1) \geq x$

$C > 0$



(5) $y' = \sqrt{1-y^2} \geq 0$ neklesající funkce

- $J = \mathbb{R}$
- $y_0 = \pm 1$ stacionární řešení
- $J_1 = (-1, 1)$ pro $|y| \geq 1$ $\sqrt{1-y^2} \leq 0$

- $$\frac{y'}{\sqrt{1-y^2}} = 1$$

$\arcsin y = x + c$ $\qquad$ $\arcsin y \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

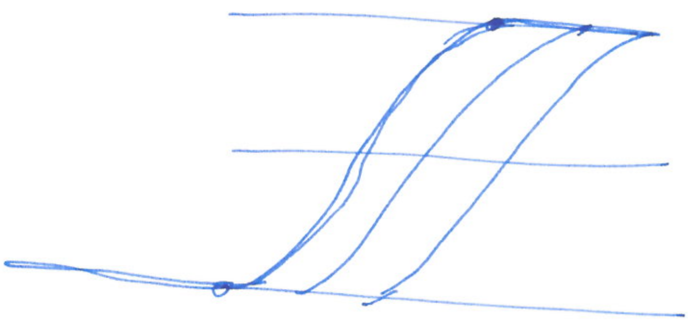
$x + c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **tady jsou imo trochu pomotaná znaménka**

$y = \sin(x+c)$ $\qquad$ $x \in \left(-\frac{\pi}{2} - c, \frac{\pi}{2} + c\right)$

- maximální řešení - zkusíme slepit se stacionárním řešením nebo řešením v jiné vrstvě

$$y(x) = \begin{cases} -1 & x \in -\frac{\pi}{2} - c \\ \sin(x+c) & x \in \left(-\frac{\pi}{2} - c, -\frac{\pi}{2} + c\right) \\ 1 & x \geq +\frac{\pi}{2} - c \end{cases}$$

- s jiným netriviálním řešením nelze
$\sqrt{1-y^2} > 0$ - vedou se nerovnající limity
nebo $y' < 0$



⑥ $y' = \frac{y \ln y}{\sin x}$ $\quad \sin x \neq 0 \quad x \neq k\pi$
$y > 0$

- $I_k = (k\pi, (k+1)\pi)$
- stacionární řešení $y_0 = 0$ a $y_0 = 1$
- $J_1 = (0,1)$, $J_2 = (1,\infty)$
- $\int \frac{1}{y \ln y}\, dy = \left| \begin{matrix} \ln y = t \\ \frac{1}{y} = \frac{dt}{dy} \end{matrix} \right| = \int \frac{1}{t} dt = \ln|t|$

$\int \frac{1}{\sin x}\, dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \frac{\cos\frac{x}{2}}{\cos\frac{x}{2}}\, dx = \frac{1}{2}\int \frac{1}{\operatorname{tg}\frac{x}{2}} \frac{1}{\cos^2\frac{x}{2}}\, dx = \left| \begin{matrix} \operatorname{tg}\frac{x}{2} = u \\ \frac{1}{2}\frac{1}{\cos^2\frac{x}{2}} = \frac{du}{dx} \end{matrix} \right|$

$= \int \frac{1}{u}\, du = \ln|u| = \ln|\operatorname{tg}\frac{x}{2}|$

- $y > 1$

$\ln|\underbrace{\ln y}_{>0}| = \ln|\operatorname{tg}\frac{x}{2}| + C$

$\ln \ln y = \ln|\operatorname{tg}\frac{x}{2}| + C$

$\ln y = e^C |\operatorname{tg}\frac{x}{2}| = k|\operatorname{tg}\frac{x}{2}|$

$y = e^{k|\operatorname{tg}\frac{x}{2}|}$

? Jak by ~~vypadalo~~ vypadalo řešení

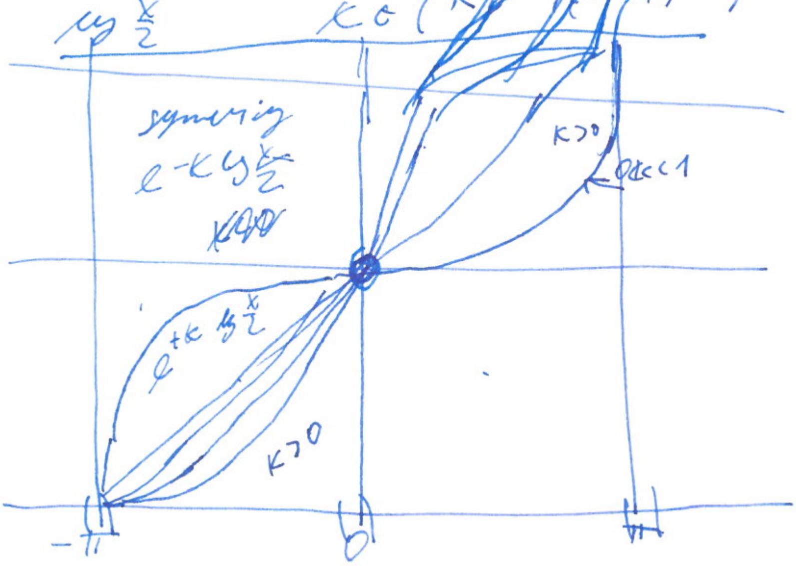
$\sin y' = y \ln y$?

- $0 < y < 1$ $\quad \ln(-\ln y) = \ln|\operatorname{tg}\frac{x}{2}| + C$

$\ln y = -e^k |\operatorname{tg}\frac{x}{2}|$

$y = e^{-k|\operatorname{tg}\frac{x}{2}|}$

$\operatorname{tg}\frac{x}{2} \quad x \in (k\pi, (k+1)\pi)$



$\frac{x}{2} \in (k\frac{\pi}{2}, k+1\frac{\pi}{2})$
$k = 0 \quad (0, \frac{\pi}{2})$
$k = -1 \quad (-\frac{\pi}{2}, 0)$
$\to k$ sudé $\quad \operatorname{tg}\frac{x}{2} > 0$
k liché $\quad \operatorname{tg}\frac{x}{2} < 0$

k sudé $\quad y > 1$
$y = e^{k\operatorname{tg}\frac{x}{2}}$
k liché $\quad y > 1$
$y = e^{-k\operatorname{tg}\frac{x}{2}}$
k sudé $\quad 0 < y < 1$
$y = e^{-k\operatorname{tg}\frac{x}{2}}$
k liché $\quad 0 < y < 1$
$y = e^{k\operatorname{tg}\frac{x}{2}}$

• pokus o navázání

→ nelze, neboť $x \neq k\pi$

→ pokud bychom uvažovali problém

$$\sin x \, y' = y \ln y$$

mohli bychom se pokusit o navázání pro sudé $|k|\pi \rightarrow \underline{2k\pi}$

(7) $y' = -2x \frac{\sqrt{1-y^2}}{y}$ $\quad y \neq 0$ a $|y| \le 1$

- $I = \mathbb{R}$ $\quad f(x) = -2x$
- $g(y) = \frac{\sqrt{1-y^2}}{y}$ $\quad g(\pm 1) = 0 \Rightarrow y_0 = \pm 1$ stac. řešení
- $J_1 = (-1,0)$ a $J_2 = (0,1)$
- $\int \frac{y}{\sqrt{1-y^2}}\, dy = -2\int x\, dx + C$ $\qquad 1-y^2 = t$, $\; -2y = \frac{dt}{dy}$

$-\frac{1}{2}\int \frac{1}{t^{1/2}} = -t^{1/2} = -x^2 + C$

$\sqrt{1-y^2} = x^2 - C$

pro J_2: $\sqrt{1-y^2} > 0 \Rightarrow x^2 - C > 0$ $\qquad |x| > \sqrt{C}$ $\qquad C \ge 0$

dále

$1-y^2 = (x^2-C)^2$

$y^2 = 1-(x^2-C)^2$ a tedy

$1-(x^2-C)^2 > 0$, abychom mohli provést inverzi

$1 > (x^2-C)^2$ víme $x^2-C > 0$ a tedy lze vzít odmoc. z $\oplus \sqrt{(x^2-C)^2}$

$1 \ge x^2 - C$

$1+C > x^2$

$\sqrt{1+C} > |x|$ $\qquad$ celkově $\mathcal{D}$: maximální řešení

$x \in (-\sqrt{1+C}, -\sqrt{C})$ a

$x \in (\sqrt{C}, \sqrt{1+C})$ maximální řešení

→ KONTROLA

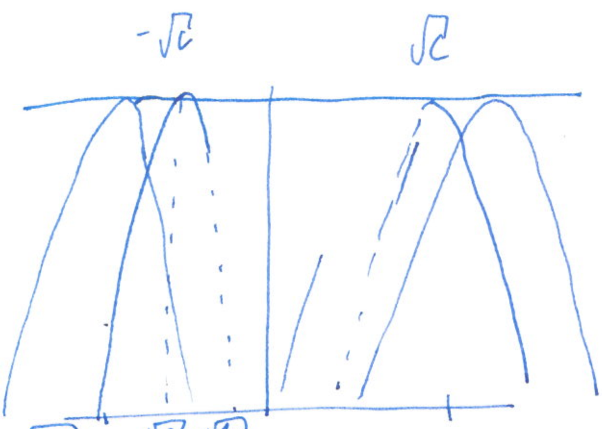
J_2 $\; y > 0$ i $\sqrt{1-y^2} > 0$

$y' = -2x \underbrace{\frac{\sqrt{1-y^2}}{y}}_{>0}$

znaménko derivace určí

$x < 0$ $\; y' > 0$ rostoucí

$x > 0$ $\; y' < 0$ klesající



symetricky

$x < -\sqrt{C}$ $\; y =$

$y > 0$ celkem: $y(x)$ $\begin{cases} -\sqrt{C} \le x < +\sqrt{C} \\ x > \sqrt{C} \end{cases}$

celkem

$y > 0$

$$y(x) = \begin{cases} \sqrt{1-(x^2-c)^2} & -\sqrt{1+c} \le x \le \sqrt{c} \\ -1 & -\sqrt{c} \le x \le \sqrt{d} \\ \sqrt{1-(x^2-d)^2} & \sqrt{1+d} \ge x \ge \sqrt{d} \end{cases}$$

rostoucí (first case)

← klesající (third case)

nebo $y < 0$

$$y(x) = \begin{cases} -\sqrt{1-(x^2-c)^2} & -\sqrt{1+c} \le x \le \sqrt{c} \\ -1 & -\sqrt{c} \le x \le \sqrt{d} \\ -\sqrt{1-(x^2-d)^2} & \sqrt{d} \le x \le \sqrt{1+d} \end{cases}$$

← klesající (first case)

← rostoucí (third case)

(8) $y' \cot g\, x + y = 2 \qquad y\left(\frac{\pi}{4}\right) = 1$

$x \neq k\pi \rightarrow x \in (k\pi, (k+1)\pi)$

- máme s počáteční podmínky
víme, že nás zajímá pouze $k = 0$
a $I = (0, \pi)$

$f(x) = \frac{1}{\cot g\, x} \qquad I_1 = (0, \frac{\pi}{2}) \quad I_2 = (\frac{\pi}{2}, \pi)$

- $g(y) = 2 - y \qquad g(2) = 0 \Rightarrow y_0 \equiv 2$

- $$\int \frac{1}{2-y}\, dy = \int \frac{1}{\cot g\, x}\, dx$$

$$\int \frac{1}{2-y}\, dy = -\ln|2-y|$$

$$\int \frac{\sin x}{\cos x}\, dx = \left| \begin{array}{l} \cos x = t \\ -\sin x = \frac{dt}{dx} \end{array} \right| = -\int \frac{1}{t}\, dt = -\ln|\cos x| + c$$

$$-\ln|2-y| = -\ln|\cos x| + \tilde{c}$$
$$\ln|2-y| = \ln|\cos x| + c$$

- $y < 2$

$\ln(2-y) = \ln|\cos x| + c \qquad c \in R$

$(2-y) = e^c|\cos x| = k|\cos x| \qquad k \geq 0$

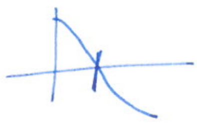
$y = 2 - k|\cos x|$

$y > 2$

$\ln(y-2) = \ln|\cos x| + c \qquad c \in R$

$y - 2 = e^c|\cos x| \qquad k \geq 0$

$y = 2 + k|\cos x|$



→ ~~...~~
nemusíme hledat všechna řešení, ale pouze takové $\cos x > 0$ na tomto intervalu

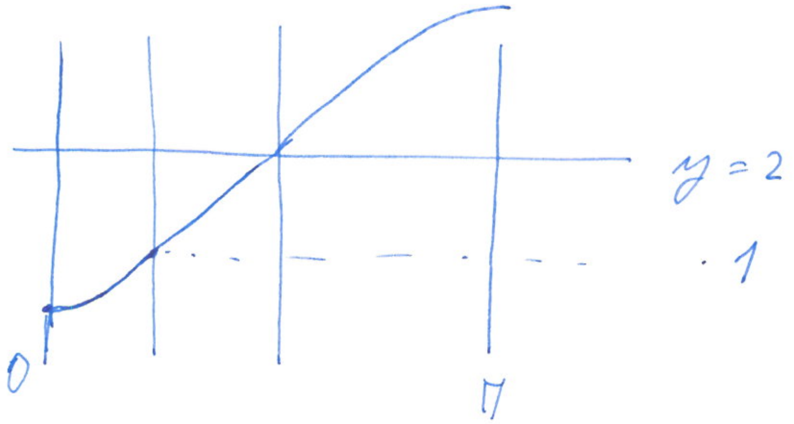
$y(\frac{\pi}{4}) = 1$

$y < 2$ a $x < \frac{\pi}{2} \qquad y = 2 - k|\cos x| = 2 - k\cos x$

$y(\frac{\pi}{4}) = 1 \rightarrow \quad 1 = 2 - k\cos\frac{\pi}{4} = 2 - k\frac{\sqrt{2}}{2}$

$k = -\sqrt{2}$

napojení v $y=2$?



$2-\sqrt{2}\cos x \quad 0<x<\frac{\pi}{2}$

v $x=\frac{\pi}{2}$ lze napojit na $2+\sqrt{2}\cos x$

$$y=\begin{cases} 2-\sqrt{2}\cos x & 0<x\le\frac{\pi}{2} \\ 2+\sqrt{2}\cos x & \frac{\pi}{2}<x<0 \end{cases}$$

$\lim_{x\to\frac{\pi}{2}^-} y(x)=2$ $\quad \lim_{x\to\frac{\pi}{2}} y'(x)=-$ $\boxed{y=2-\sqrt{2}\cos x}$

$\lim_{x\to\frac{\pi}{2}^+} y(x)=2$

$y>2$ $\quad$ ~~$y=2+\tilde{k}|\cos x|$~~ chceme mít derivaci

$y=2+\tilde{k}|\cos x| \qquad \cos x<0$

$y=2-\tilde{k}\cos x$

$y'=+\tilde{k}\sin x \qquad v\ \frac{\pi}{2}\to\tilde{k}$

chceme spojitou a dostaneme $\tilde{k}=-\sqrt{2}$

→ řešení na celém intervalu $y=2-\sqrt{2}\cos x$
$(0,\pi)$

(9) $y' = -\frac{x}{y}\,\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ $|x|<1$

$|y|\le 1$ a $y\neq 0$

• $f(x) = -\frac{x}{\sqrt{1-x^2}}$ $I=(-1,1)$

• $g(y) = \frac{\sqrt{1-y^2}}{y}$ $g(\pm 1)=0 \to y_0 = \pm 1$

• $J_1 = (-1,0)$ a $J_2 = (0,1)$

• znaménko derivace : $y>0$ $x<0$ rostoucí $y'>0$

$x>0$ klesající $y'<0$

$\int \frac{y}{\sqrt{1-y^2}} = -\int \frac{x}{\sqrt{1-x^2}} + \tilde{c}$

$-\sqrt{1-y^2} = \sqrt{1-x^2} + \tilde{c}$ $\tilde{c} = -c$

$\underbrace{\sqrt{1-y^2}}_{\ge 0} = c - \sqrt{1-x^2}$

$\Rightarrow c-\sqrt{1-x^2}\ge 0$

$c \ge \underbrace{\sqrt{1-x^2}}_{\ge 0} \Rightarrow \boxed{c\ge 0}$

a $\boxed{|x| \ge \sqrt{1-c^2}}$

$1-y^2 = (c-\sqrt{1-x^2})^2$

$y^2 = 1-(c-\sqrt{1-x^2})^2$

pro $1-(c-\sqrt{1-x^2})^2 > 0$

$\boxed{y = \pm\sqrt{1-(c-\sqrt{1-x^2})^2}}$

$1-(\underbrace{c-\sqrt{1-x^2}}_{\ge 0})^2 > 0$

$1 \ge |c-\sqrt{1-x^2}|$

$\sqrt{1-x^2} \ge c-1$

$\Rightarrow$ splněno vždy pro $c<1$

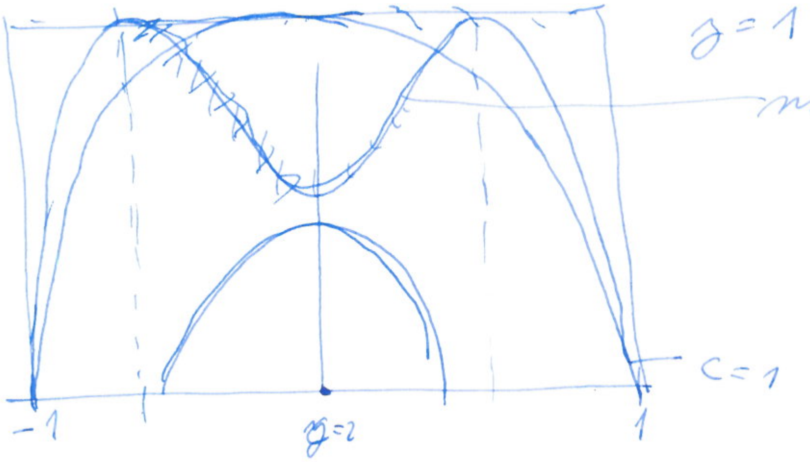
$\Rightarrow c>1$ $1-x^2 > (c-1)^2$

$1-(c-1)^2 > x^2$

$2c-c^2 > x^2$

$\sqrt{c}\sqrt{2-c} > |x| \Rightarrow c\le 2$

a pro $c>1$ $\boxed{|x|<\sqrt{2c-c^2}}$



po goavne
$|x| \geq \pm\sqrt{1-c^2}$

$$1 < c < 2$$

$$y = \pm\sqrt{\cancel{4}\left(c-\sqrt{1-x^2}\right)^2} \qquad x < -\sqrt{1-c^2}$$

$$-\sqrt{1-c^2} < x < \sqrt{1-c^2}$$

$$y = 1$$

(10) $y' = \frac{\sqrt{y^2+1}}{xy}$ $\qquad x \neq 0$, $y \neq 0$

- $f(x) = \frac{1}{x}$ $\quad I_1 = (-\infty, 0)$ $\quad I_2 = (0, \infty)$
- $g(y) = \frac{\sqrt{y^2+1}}{y}$ $\quad$ nemá stacionární řešení
- $J_1 = (-\infty, 0)$, $J_2 (0, \infty)$

$I_{1,2} \times J_{1,2}$ – rozumné

$I_1 \times J_2$ a $I_2 \times J_1$ zrcadlí

$$\int \frac{y}{\sqrt{y^2+1}}\, dy = \sqrt{y^2+1} = \ln|x| + C$$

$\ln|x| + C > 0$

$\ln|x| > -C$ $\qquad x \in (-\infty, -e^{-C})$

$|x| > e^{-C}$ $\qquad$ nebo $x \in (e^{-C}, \infty)$

dále můžeme pokračovat

$y^2 = (\ln|x| + C)^2 - 1$

$(\ln|x| + C)^2 \geq 1$

pak $\boxed{y = \pm \sqrt{(\ln|x| + C)^2 - 1}}$

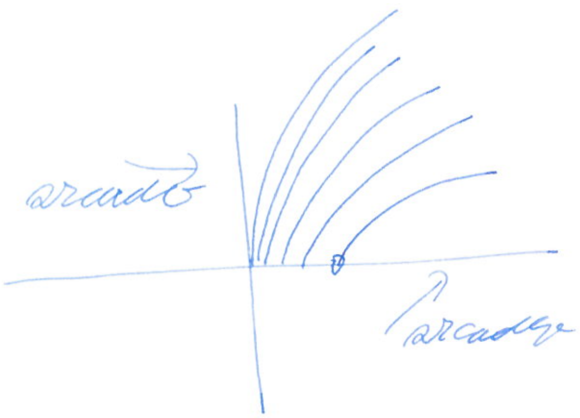
$\ln|x| + C > 0$ již víme

$\ln|x| + C > 1$

$|x| = e^{1-C}$

$x \in (-\infty, -e^{1-C})$

$x \in (e^{1-C}, \infty)$



to je x, ne +

(11) $y' = \frac{2x y^2}{1-x^2}$ $\quad y(0)=1 \quad x \neq \pm 1$

- $f(x) = \frac{2x}{1-x^2}$ $\quad I_1 = (-\infty, -1)$, $\underbrace{I_2 = (-1, 1)}$, $I_3 = (1, \infty)$

zajímá nás
v krajních bodech
nemůžeme navázat

- $g(y) = y^2$ $\quad g(0) = 0 \Rightarrow y_0 = 0$ triviální řešení
- $J_1 = (-\infty, 0)$, $J_2 = (0, \infty)$
- $\int \frac{1}{y^2}\,dy = \int \frac{x}{1-x^2}\,dx + C$

$-\frac{1}{y} = -\ln|1-x^2| + C$

$y(0) = 1$

$-1 = -\ln 1 + C \Rightarrow C = -1$

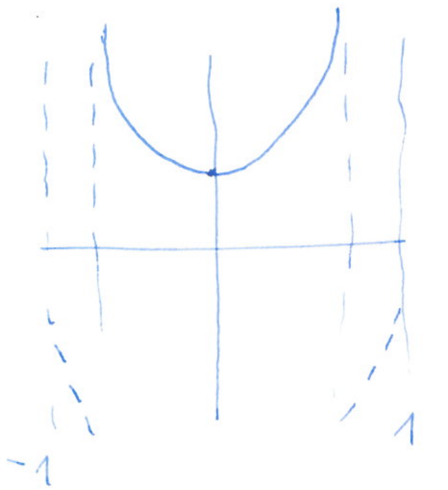
$\frac{1}{y} = \ln(1-x^2) + 1$

$\ln(1-x^2) \neq -1 \qquad 1-x^2 \neq e^{-1}$

$1-e^{-1} \neq x^2$

$x \neq \pm\sqrt{1-e^{-1}}$

$y = \frac{1}{\ln(1-x^2)+1} \qquad x \in \left(-\sqrt{1-\frac{1}{e}}, \sqrt{1-\frac{1}{e}}\right)$



(12) nalezněte všechna maximální řešení

$y'(2-e^x) = -3e^x \, tg\, y \cos^2 y$ $\qquad$ $\cancel{y \neq k\frac{\pi}{2}}$

$y \neq k\pi + \frac{\pi}{2}$

a) $y(\ln 3) = 0$

b) $y(\ln 3) = \frac{\pi}{4}$

c) $y(\ln 3) = \frac{\pi}{2}$

• $f(x) = -\frac{3e^x}{2-e^x}$ $\qquad$ $I_1 = (-\infty, \ln 2)$

$I_2 = (\ln 2, \infty)$

$2 = e^x$

$\ln 2 = x$

• $g(y) = tg\, y \cos^2 y$

$g(y) = 0$ $\qquad tg\, y = 0 \qquad y = k\pi$ $\qquad \boxed{y_0 = k\pi}$

$\cos^2 y = 0 \qquad \frac{\pi}{2} + k\pi \to$ není z Def. oboru

navíc více $\quad y = 0,\ \frac{\pi}{4},\ \frac{\pi}{2}$ $\to$ zajímavé intervaly

$I_1 = (-\frac{\pi}{2}, 0)$

•

$I_2 = (0, \frac{\pi}{2})$

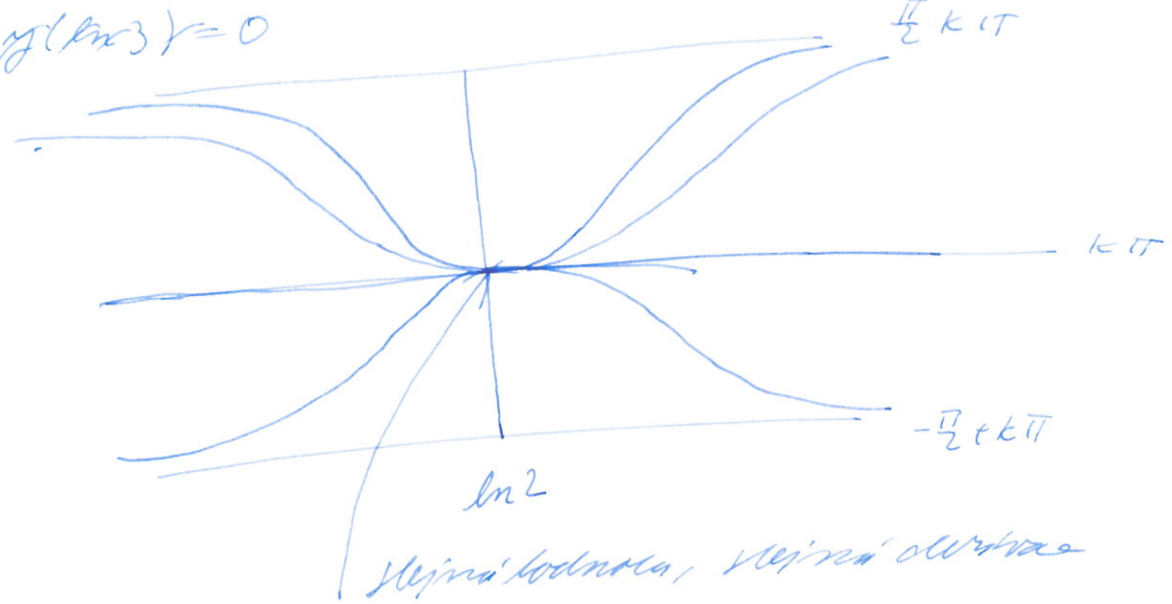
• $\int \frac{dy}{tg\, y \cos^2 y} = -3 \int \frac{e^x}{2-e^x} dx + \tilde{c}$

$\ln |tg\, y| = c\,|2-e^x|^3$

$tg\, y = sign(tg\, y)\, c\, |2-e^x|^3$

$y = k\pi + atan\left(sign(tg\, y)\, c\, |2-e^x|^3\right)$

a) $y(\ln 3) = 0$



stejná hodnota, stejná derivace

a) $y(\ln 3) = 0$

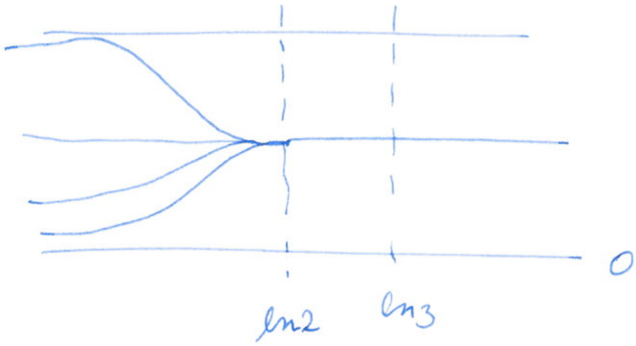
$|\operatorname{tg} y| = c\,|2 - e^x|^3$

$0 = c\,|2-3|^3 \to c = 0$

$y = \operatorname{arctan}(\operatorname{sign} y_0)\, c\,|2-e^x|^3$ pro $x < \ln 2$

$y = 0 \quad x > \ln 2$

nebo $y \equiv 0$



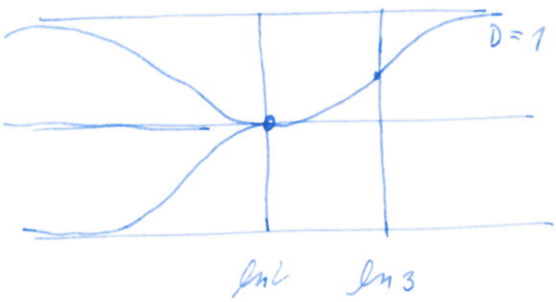
b) $y(\ln 3) = \frac{\pi}{4}$

$|\operatorname{tg}\frac{\pi}{4}| = c\,|2 - e^{\ln 3}|^3$

$1 = c\,|2-3|^3 = c$

$1 = c$

$$y = \begin{cases} \operatorname{arctan}(\operatorname{sign} y_0)\, c\,|2-e^x|^3 \\ \operatorname{arctan} D(e^x-2)^3 \end{cases}$$



c) nelze

(13) $xy' - y = 0$ všemi body $x=0, y=0$

$f(x) = \frac{1}{x}$, $g(y) = y$

· $I_1 = (-\infty, 0)$, $I_2 = (0, \infty)$

· $g(0) = 0$ $y_0 = 0$

· $\frac{y'}{y} = \frac{1}{x}$, $J_1 = (-\infty, 0)$, $J_2 = (0, \infty)$

$\int \frac{1}{y}\, dy = \int \frac{1}{x}\, dx + c \quad c \in \mathbb{R}$

$|y| = k|x| \quad k \in (0, \infty)$

$y = \tilde{k} x \quad \tilde{k} \in \mathbb{R}$

(13) $xy' - y = 0$ řešení kromě $x=0, y=0$

· $f(x) = \frac{1}{x}$ $I_1 = (-\infty, 0),\ I_2 = (0, \infty)$

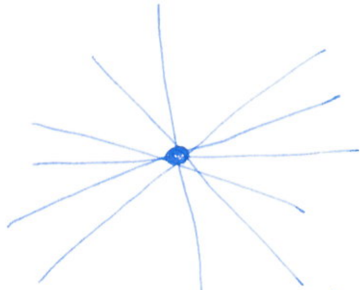
· $g(x) = y$ $g(0) = 0$

· $J_1 = (-\infty, 0),\ J_2 = (0, \infty)$

· $\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx + C$

$|y| = k|x|$

$y = \tilde{k}x$ – nelze navázat ~~e~~ dr. spojení sp. přes 0 – ok



(14) $\frac{d^2 r}{dt^2} = -\frac{GM}{r^2} m$ $\mu = \frac{Mm}{M+m} \approx m$ ($M \gg m$)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$ substituce

MOŽNOST 1

$\frac{d}{dt}\left(\frac{dr}{dt}\right)^2 = \underline{2\frac{dr}{dt}\frac{d^2 r}{dt^2}} = -2\frac{GM}{r^2}\frac{dr}{dt}$ $\int_0^1 ds$

$\left(\frac{dr}{dt}(t_0)\right)^2 - 0 = +2GM\left[\frac{1}{r}\right]_h^{R_Z} = 2GM\left(\frac{1}{r} - \frac{1}{h}\right)$

$\frac{dr}{dt} = -\sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{h}} =$

$\frac{dr}{dt} = -\sqrt{2GM}\sqrt{\frac{1}{r}\left(1 - \frac{r}{h}\right)}$

$\frac{r}{h}$

$\int_h^{R_Z} \frac{\sqrt{h}\sqrt{\frac{r}{h}}}{\sqrt{1 - \frac{r}{h}}}\,dr = -\sqrt{2GM}\,t$ $\left| \begin{array}{l} u = \sqrt{\frac{r}{h}} \\ \frac{du}{dr} = \frac{1}{\sqrt{h}}\frac{1}{2}\frac{1}{\sqrt{r}} \end{array} \right|$

$= h\int_1^{\sqrt{\frac{R_Z}{h}}} \frac{2u^2}{\sqrt{1-u^2}}\,du = \left| \begin{array}{l} u = \sin v \\ \frac{du}{dv} = \cos v \end{array} \right| = 2h\int_{\frac{\pi}{2}}^{\arcsin\sqrt{\frac{R_Z}{h}}} \sin^2 v\,dv$

$= 2h\int \left(\frac{1}{2} - \frac{1}{2}\cos 2v\right)dv =$

$= h\int (1 - \cos 2v)\,dv$

$= h\left[v - \frac{\sin 2v}{2}\right]_{\frac{\pi}{2}}^{\arcsin\sqrt{\frac{R_Z}{h}}}$

$= h\left(\frac{\pi}{2} - \arcsin\sqrt{\frac{R_Z}{h}} - \frac{\sin 2\sqrt{\frac{R_Z}{h}}}{2}\right) = h - R_Z$

$h \to \infty$

$$-\frac{GM}{r^2} = \frac{d^2 r}{dt^2}$$ MOŽNOST 2

$$\frac{d^2 r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

$$-\frac{GM}{r^2} = v\frac{dv}{dr}$$

$$\frac{GM}{r} + \tilde{a} = \frac{v^2}{2}$$

$$\pm\sqrt{2\frac{GM}{r} + a} = v = \frac{dr}{dt}$$

$$-\int \frac{dr}{\sqrt{2\frac{GM}{r} + a}} = \int dt + v$$

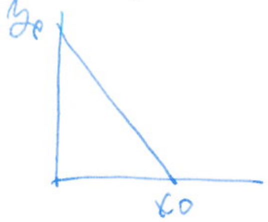
$$\cancel{\frac{\sqrt{r}}{\sqrt{2GM}\sqrt{1+a}}} \quad \frac{1}{\sqrt{2\frac{GM}{r} + a}} = \frac{\sqrt{r}}{\sqrt{2GM + ar}} = \frac{\sqrt{r}}{\sqrt{a}\sqrt{2\frac{GM}{a} + r}} =$$

$$= \frac{\sqrt{r}}{\sqrt{a}} \frac{1}{\sqrt{c + r}}$$

$$\int \frac{\sqrt{r}}{\sqrt{c+r}}\,dr = \left| \begin{matrix} u = \sqrt{r} \\ \frac{du}{dr} = \frac{1}{2}\frac{1}{\sqrt{r}} \end{matrix} \right| = \int \frac{2u^2}{\sqrt{c + u^2}} = \left| \begin{matrix} u = c\sin v \\ \frac{du}{dv} = c\cos v \end{matrix} \right| =$$

$$= 2\int \frac{c^2 \sin^2 v}{c\sqrt{1 - \cos^2 v}} \quad \rightarrow$$

15. ? $\cancel{y' = -tg\frac{y}{x}}$



$\frac{x_0}{2} = x \quad \rightarrow 2x = x_0$

$y = y_0 + k(x - x_0)$ v $x = x_0$ $y = 0$

a tedy $y = k(x - x_0)$

dále víme $k = y'$ a $x_0 = 2x$

a tedy $y = -y'x$

$\ln|y| = -\ln|x| + C$

$|y| = Ke^{-\ln|x|} = \frac{K}{|x|}$

$y = \frac{K}{|x|}$

x, r

$(x, y) = (2, 3) \rightarrow K = 6$

$y = \frac{6}{x}$

(14) pohybová rovnice

$$\mu \frac{d^2r}{dt^2} = -\frac{GM}{r^2} m \qquad \mu = \frac{mM}{m+M} \overset{M \gg m}{\approx} m$$

$$\Rightarrow \frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

MOŽNOST 1

derivace kinetické energie $\sim v^2$

$$\frac{d}{dt}(v^2) = 2v\frac{dv}{dt} = 2\frac{dr}{dt}\frac{d^2r}{dt^2} = -2\frac{GM}{r^2}\frac{dr}{dt}$$

$$\left(\frac{dr}{dt}\right)^2(t) - \underbrace{\left(\frac{dr}{dt}\right)^2(t=0)}_{\substack{=0 \\ poč.\ podmínka}} = 2GM\left(\frac{1}{r} - \frac{1}{h}\right) = \frac{h-r}{rh}2GM$$

vyvolíte

$$\frac{dr}{dt} = (-)\sqrt{\frac{h-r}{rh}}\sqrt{2GM}$$

MOŽNOST 2

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

$$-\frac{GM}{r^2} v \frac{dv}{dr}$$

$$\frac{GM}{r} - \tilde{a} = \frac{v^2}{2} \qquad a \to h$$

$$\frac{dr}{dt} = -\sqrt{2GM}\sqrt{\frac{1}{r} - a} \to \frac{1}{h} \Rightarrow \frac{1}{\sqrt{\frac{1}{r}-\frac{1}{h}}}\frac{dr}{dt} = -\sqrt{2GM}$$

S1 $\sqrt{\frac{1}{r}-\frac{1}{h}} = u$

S2 $\sin \alpha = u$

pro $h \to \infty$

$$\frac{1}{\sqrt{\frac{1}{r}}}\frac{dr}{dt} = -\sqrt{2GM}$$

$$\frac{2}{3}r^{3/2} = \sqrt{2GM}(t - t_0)$$

STAČÍ RYCHLOST!!

$$v = -\sqrt{2GM}\sqrt{\frac{1}{r}-\frac{1}{h}} \qquad r = R_Z$$

$$v = -\sqrt{2GM}\sqrt{\frac{1}{R_Z}-\frac{1}{h}}$$

$$h \to \infty \quad v = -\sqrt{\frac{2GM}{R_Z}}$$

ZZE: $\frac{1}{2}mv_h^2 - \frac{GM}{h} = \frac{1}{2}mv^2 - \frac{GM}{r} \Rightarrow$ stejné odvození