① $y' \cos x = y \sin x + \cos^2 x$

→ lin. rovnice $y = y_H + y_P$

$y_H' \cos x - y_H \sin x = 0$ $\quad y_H \neq 0,\ \cos x \neq 0$

$\ln|y_H| = -\ln|\cos x| + \tilde{\tilde{C}}$

$|y_H| = \tilde{C} e^{-\ln|\cos x|}$

$y_H = C \frac{1}{|\cos x|} = \frac{C}{|\cos x|} = \frac{C}{\cos x}$ $C \in R$, na podintervalu $y_H = \frac{C}{\cos x}$

pro $\cos x = 0$ $\quad y \sin x = 0 \rightarrow y = 0$

$x \in \left(\frac{\pi}{2} + k\pi, \frac{3}{2} + k\pi\right)$

C závisí na poč. podmínce

$y_P = \frac{C(x)}{\cos(x)}$

$y_P' = \frac{C'(x)}{\cos x} + \frac{C(x)}{\cos^2 x} \sin x$

$C'(x) + C(x) \frac{\sin x}{\cos x} = \frac{C(x)}{\cos x} \sin x + \cos^2 x$

$C'(x) = \cos^2 x$

$C = \frac{1}{2} \int (\cos 2x + 1)\, dx =$

$= \frac{1}{2} \left(\frac{\sin 2x}{2} + x \right) = \frac{2 \sin x \cos x}{4} + \frac{x}{2} =$

$= \frac{1}{2} (\sin x \cos x + x)$

celkově $y_P = \frac{\frac{1}{2} (\sin x \cos x + x)}{\cos x}$

$y = y_H + y_P = \frac{C}{\cos x} + \frac{1}{2} \frac{1}{\cos x} (\sin x \cos x + x)$

② $y' - 2\frac{y}{x} = x^3$ $\qquad x \neq 0$

$\rightarrow y = y_H + y_P$

$\rightarrow y_H$:

$y_H' - 2\frac{y_H}{x} = 0 \rightarrow y_H' = \frac{2}{x} y_H \qquad y_H \equiv 0$ triviální ~~[illegible]~~

$\ln|y_H| = 2\ln|x| + \tilde{c} \qquad \tilde{c} \in \mathbb{R}$

$|y_H| = \tilde{c} e^{\ln|x|^2} \qquad \tilde{c} \in (0, \infty)$

$y_H = c x^2 \qquad c \in \mathbb{R}$, $c = 0$ triv. řeš.

$\rightarrow y_P$

$y_P = c(x) x^2$

$y'(P) = c'(x) x^2 + 2c(x) x$

$c'(x) x^2 + 2c(x) x - 2cx = x^3$

$c'(x) = x$

$c = \frac{x^2}{2}$

$y_P = \frac{x^4}{2}$

$\rightarrow y = y_H + y_P = cx^2 + \frac{x^4}{2}$

(3) $y' + 2xy = 2x e^{-x^2}$

$\to y = y_H + y_P$

$\to y_H \quad y_H' + 2x y_H = 0$

$y_H \neq 0 \quad \ln|y_H| = -x^2 + \tilde{c} \qquad \tilde{c} \in \mathbb{R}$

$|y_H| = e^{\tilde{c}} e^{-x^2} = \tilde{\tilde{c}} e^{-x^2} \qquad \tilde{\tilde{c}} \in (0, \infty)$

$y_H = c\, e^{-x^2} \qquad c \in \mathbb{R}$

$c = 0$ triv. řešení

$\to y_p = c(x) e^{-x^2}$

$y_p' = c'(x) e^{-x^2} - 2x e^{-x^2} c(x)$

$y_p' + 2xy = 2x e^{-x^2}$

$c'(x) e^{-x^2} = 2x e^{-x^2}$

$c = x^2$

$y_p = x^2 e^{-x^2}$

$\to y = y_H + y_p = c\, e^{-x^2} + x^2 e^{-x^2} = e^{-x^2}(c + x^2)$

(4) $y' + y \sin x = \sin x \cos x$

$\rightarrow y = y_p + y_H$

$\rightarrow y_H' + y_H \sin x = 0 \qquad y_H \neq 0$

$$\ln|y_H| = \cos x + \tilde{c} \qquad \tilde{c} \in \mathbb{R}$$

$$|y_H| = \hat{c}\, e^{\cos x} \qquad \hat{c} = e^{\tilde{c}} \qquad \hat{c} \in (0, \infty)$$

$$y_H = c\, e^{\cos x} \qquad c \in \mathbb{R}\text{, } c = 0 \text{ triviální}$$

$\rightarrow y_p = c(x)\, e^{\cos x}$

$$y'(x) = c'(x)\, e^{\cos x} - \sin x\, c\, e^{\cos x}$$

$$c'(x) = \sin x \cos x\, e^{-\cos x}$$

$$c(x) = \int \cos x \left(\sin x\, e^{-\cos x}\right) dx = \left| \begin{array}{ll} f'(x) = \sin x\, e^{-\cos x} & f = e^{-\cos x} \\ g = \cos x & f' = -\sin x \end{array} \right|$$

$$= \cos x\, e^{-\cos x} + \int \sin x\, e^{-\cos x}\, dx = \cos x\, e^{-\cos x} + e^{-\cos x}$$

$$y_p = c(x)\, e^{\cos x} = \cos x + 1$$

$\rightarrow y = y_H + y_p = c e^{\cos x} + \cos x + 1$

⑤ $xy' + y = \ln x + 1$ $\qquad x > 0$

$\to y = y_p + y_H$

$x y_H' = -y$

$\ln|y_H| = -\ln|x| + \tilde{C}$

$y_H = \frac{C}{x}$

$\to y_p = \frac{C(x)}{x}$

$\to \quad x \frac{C'(x)}{x} = \ln x + 1$

$C'(x) = \ln x + 1$

$C(x) = x\ln x - x + x =$

$y_p = \frac{x}{x} \ln x = \ln x$

$\to y = \frac{C}{x} + \ln x$

⑥ $(2e^y - x)y' = 1$ $\quad$ pro $y' \neq 0$ $\quad x' = \frac{1}{y'}$

blackpenova metoda, "převrátím"

$x' = 2e^y - x$

$\to x = x_p + x_H$

$\to x_H: \quad \ln|x_H| = -y + \tilde{C} \quad\quad x_H \neq 0$

$|x_H| = \tilde{C} e^{-y}$

$x_H = Ce^{-y} \quad \to C \in \mathbb{R}$, $C = 0$ pro $x_H = 0$

$\to x_p = c(y)e^{-y}$

$c'(y) = 2e^{2y}$

$c(y) = e^{2y}$

$x_p = e^{2y}e^{-y} = e^y$

$\to x = x_p + x_H = Ce^{-y} + e^y$

(2) $y' \sin 2x = 2(y + \cos x)$

$\to y = y_H + y_p$

$\to y' \sin 2x = 2y$ $\qquad y = 0$ triviální řešení

$\frac{y'}{y} = \frac{2}{\sin 2x}$ pro $y \neq 0$ a $\sin 2x \neq 0$, $x \neq \frac{\pi}{2} k$

$\ln|y| = \int \frac{1}{\sin x \cos x} \frac{\cos x}{\cos x} dx + C = \ln|\operatorname{tg} x| + C$

$y = C \operatorname{tg} x$

$\to y_p = C(x) \operatorname{tg}(x)$

$\sin 2x \operatorname{tg} x \, C'(x) = 2 \cos x$

$2 \sin x \cos x \frac{\sin x}{\cos x} C' = 2 \cos x$

$C' = \int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}$

$\to y = y_H + y_p = C \operatorname{tg} x - \operatorname{tg} x \frac{1}{\sin x} = C \operatorname{tg} x - \frac{1}{\cos x}$

$y \cos x = C \sin x - 1$

$\to$ hledáme, aby bylo řešení konečné pro $\frac{\pi}{2}$

$L = \lim_{x \to \frac{\pi}{2}} \frac{C \sin x - 1}{\cos x} = \left| \begin{matrix} t = x - \frac{\pi}{2} \\ x = t + \frac{\pi}{2} \end{matrix} \right| = \lim_{t \to 0} \frac{C \sin(t + \frac{\pi}{2}) - 1}{\cos(t + \frac{\pi}{2})}$

$= -\lim_{t \to 0} \frac{C \cos t - 1}{\sin t} = -\lim_{x \to 0} \frac{C(1 - \frac{t^2}{2}) - 1}{t}$

pro $C = 1$ $L = 0$ jinak $L \to \infty \Rightarrow C = 1$

$y \cos x = \sin x - 1$

(8) $xy' \cancel{=} 2x^2\sqrt{y} = 4y$ $y>0$ $x \neq 0$ $\boxed{\text{pro } x=0 \quad y=0}$

$y' - \frac{4}{x}y = 2x\sqrt{y}$ $\alpha = 1/2$ místo ~~proměnná~~ navážem s triviálním řešením

$z(x) = y^{1-\alpha}(x) = \sqrt{y(x)}$ $z = \sqrt{y}$

$z' = \frac{1}{2}\frac{1}{\sqrt{y}}y' = \frac{1}{2}\frac{y'}{z} \rightarrow y' = 2zz'$

~~$2xzz'$~~

$2zz' - \frac{4}{x}z^2 = 2xz$ $z \neq 0$

$2z' - \frac{4}{x}z = 2x \rightarrow z' - \frac{2}{x}z = x$

$\rightarrow \cancel{z} z_H' - \frac{2}{x}z_H = 0$ $z_H \neq 0$ $z_H = 0$ triv. řeší

$\frac{z_H'}{z_H} = \frac{2}{x} \rightarrow \ln|z_H| = 2\ln|x| + \tilde{C}$

$z_H = c\,x^2$

$z_p = c(x)x^2$

Tady jsme dosadili to C(x)x^2 do té původní rovnice z′-2/x*z=x, dva členy se pokrátili

$c'(x)x^2 - \cancel{\frac{2}{x}c} = x$

$c' = \frac{1}{x}$

$c = \ln|x| \rightarrow z_p = x^2\ln|x|$

$z = cx^2 + x\ln|x|$

$\sqrt{y} = cx^2 + x\ln|x| \rightarrow y = (cx^2 + x^2\ln|x|)^2 = x^2(c+\ln|x|)^2$

~~kde~~ $x \neq 0$ ale existují limity

~~$y = x^2(c+\ln|x|)^2$ $x<0$~~

~~$y=0$~~ ~~$x>0$~~

$\vee\ y = 0$

$y = x^4(c+\ln|x|)^2$ $x>0$

$\vee\ y=0$

(9) $y' - 2xy = 2x^3y^2$ $\quad \alpha = 2 \leftarrow$ $x=0\ y'=0$

~~nelze nav~~
není vždy navazovat

$z(x) = y^{(1-\alpha)} = \frac{1}{y}$ $\quad y \neq 0$

$y=0$ triviální

$z' = -\frac{1}{y^2}y' \rightarrow y' = -y^2z'$

$-y^2z' - 2xy = 2x^3y^2 \quad / \frac{1}{y^2}$

$-z' - 2xz = 2x^3$

$\rightarrow z = z_p + z_H$

$-z' - 2x\cancel{z} = 0$

$z' = -2xz \quad z \neq 0$

$\frac{z'}{z} = -2x$

$\ln|z| = -x^2 + \tilde{C}$

$z = ce^{-x^2}$

$\rightarrow z_p = C(x)e^{-x^2}$

$-C'(x)e^{-x^2} = 2x^3$

$C'(x) = -2e^{x^2}x^3$

$C(x) = -2\int e^{x^2}\cdot 2x\ \frac{x^2}{2}\,dx = \left| \begin{array}{l} f'(x) = e^{x^2}\cdot 2x \quad f = e^{x^2} \\ g(x) = \frac{x^2}{2} \quad g' = x \end{array} \right.$

$= -2\frac{x^2}{2}e^{x^2} + 2\frac{1}{2}\int 2x e^{x^2}\,dx =$

$= -2\frac{x^2}{2}e^{x^2} + 2\frac{1}{2}e^{x^2}$

$z_p = -\frac{x^2}{2} + \frac{1}{2}$ $\quad \rightarrow z = ce^{-x^2} + \frac{1}{2}(1-x^2)$

$\rightarrow y = +\frac{1}{ce^{-x^2} + \frac{1}{2}(1-x^2)}$ na intervalu, kde je jmenovatel nenulový

(10) $y' - \frac{1}{x}y = \frac{1}{2y}$ $\qquad x \neq 0$

$y \neq 0$

$\alpha = -1$

$z = y^{1-\alpha} = y^2$ z rovnice lineární

$\to z' = 2yy'$

$x\,2yy' - 2y^2 = x\frac{2y}{2y}$

$xz' - 2z = x$

$\to z = z_H + z_P$

$z' - 2\frac{z}{x} = 1$

$\to z_H$: $\ln|z| = 2\ln|x| + C$

$z = Cx^2$ zahrnující i triv. řešení $z = 0$

$\to z_P = C(x)x^2$

$2xC + C'x^2 - \frac{2}{x}Cx^2 = 1$

$C'x^2 = 1$

$C = \int \frac{1}{x^2}dx = -\frac{1}{x}$

$z = z_P + z_H = Cx^2 - \frac{1}{x}x^2 = x(Cx-1)$ kde $Cx^2 - x > 0$

$y = \sqrt{x(Cx-1)}$

$x(Cx-1) > 0$

$x < 0$ & $Cx - 1 < 0$

$x < \frac{1}{C}$

$x > 0$ & $Cx - 1 > 0$

$x > \frac{1}{C}$

(11) $xy' + y = y^2 \ln x \quad / y^2 \qquad y(1) = 1$

można zrobić [illegible]

$\alpha = 2 \qquad z = y^{1-\alpha} = \frac{1}{y} \,;\, z' = -\frac{1}{y^2} y' \qquad y \neq 0$

$y = 0$

$y' = 0 \quad y = 0$

$x \frac{y'}{y^2} + \frac{1}{y} = \ln x$

$-xz' + z = \ln x$

$z' - \frac{z}{x} = -\frac{1}{x} \ln x$

$\rightarrow z = z_H + z_p$

$\rightarrow z_H' - \frac{z_H}{x} = 0 \qquad \frac{z_H'}{z_H} = \frac{1}{x}$

$z_H = Cx$

$\rightarrow z_p = C(x)\, x$

$x C' \cancel{x} = -\frac{1}{x} \ln x$

$C' = -\frac{1}{x^2} \ln x$

$C' = -\frac{1}{x} \ln x \cdot \frac{1}{x}$

$C = -\int \frac{\ln x}{x} \cdot \frac{1}{x} = \left| \begin{array}{ll} f' = \frac{1}{x^2} & f = -\frac{1}{x} \\ g = \ln x & g' = \frac{1}{x} \end{array} \right|$

$= \frac{\ln x}{x} - \int \frac{1}{x^2} dx = \frac{\ln x}{x} + \frac{1}{x} = \frac{\ln x + 1}{x}$

$z_p = \ln x + 1 \qquad y(1) = 1 \rightarrow y = \frac{1}{z}$

$z(1) = 1$

$\rightarrow z = Cx + \ln x + 1$

$y = \frac{1}{Cx + \ln x + 1}$

$z(1) = C + 0 + 1 = 1 \qquad C = 0$

$x > e^{-1} \qquad y = \frac{1}{\ln x + 1}$

(12) $xy' - xy = -y^3 e^{-x^2}$

triv. řešení $\to y = 0 \to y' = 0 \to$

$x = 0 \quad y' = -y^3$

$\boxed{y=0 \text{ triviální}}$

$\alpha = 3 \qquad z = \frac{1}{y^2}$

$z' = -2\frac{1}{y^3} \cdot y'$

$-2\frac{y'}{y^3} + 2\frac{1}{y^2} = +2e^{-x^2}$

$z' + 2xz = 2e^{-x^2}$

$\to z = z_H + z_P$

$z_H' + 2xz = 0$

$\frac{z'}{z} = -2x$

$\ln|z| = -2\frac{x^2}{2} + \tilde{C}$

$|z| = \tilde{C} e^{-x^2}$

$z = Ce^{-x^2}$

$\to z_P = C(x)e^{-x^2}$

$C'e^{-x^2} = 2e^{-x^2}$

$C' = 2x$

$z_P = 2xe^{-x^2}$

$\to z = Ce^{-x^2} + 2xe^{-x^2} = e^{-x^2}(C+2x)$

$y = \pm\frac{e^{\frac{x^2}{2}}}{\sqrt{C+2x}} \qquad x > \frac{C}{2}$

(13) $y' - 9x^2 y = (x^5 + x^2) y^{2/3}$ $\qquad y(0)=0$

$x=0 \to y'=0$

$y=0 \to y'=0$

$y \equiv 0$ triviální řešení

$\alpha = \frac{2}{3} \qquad z = y^{1-\alpha} = y^{1/3}$

$z' = \frac{1}{3} y' y^{-2/3}$

$\frac{1}{3}\frac{y'}{y^{2/3}} - \frac{1}{3} 9x^2 y^{1/3} = (x^2 + x^5)\cdot\frac{1}{3}$

$z' - 3x^2 z = \frac{1}{3}(x^2 + x^5)$

$\to z = z_H + z_p$

$\to z_H: \quad z' = x^2 z \qquad z \neq 0$

$\frac{z'}{z} = x^2$

$\ln|z| = x^3 + c$

$z = c e^{x^3}$

$\to z_p = c(x) e^{x^3}$

$c' e^{x^3} = \frac{1}{3}(x^2 + x^5)$

$c' = \frac{1}{3} e^{-x^3}(x^2 + x^5)$

$c = \frac{1}{3}\int \left(e^{-x^3} x^2 + e^{-x^3}\cdot x^2 \cdot x^3\right) = \left| \begin{matrix} f'(x) = x^2 e^{-x^3} & f = \frac{e^{-x^3}}{-3} \\ g = 1 + x^3 & g' = 3x^2 \end{matrix} \right.$

$= -\frac{1}{3}\frac{1}{3}(1+x^3) e^{-x^3} + \frac{1}{3}\int e^{-x^3} x^2 =$

$= \left(-(1+x^3) e^{-x^3} - e^{-x^3}\right)\frac{1}{9} = -(2+x^3) e^{-x^3}\cdot\frac{1}{9}$

$z = c e^{x^3} - \frac{(2+x^3)}{9} \qquad y = \left(c e^{x^3} - \frac{2+x^3}{9}\right)^3$

$y(0) = 0$, navázání v nule na triv. řešení

$0 = c - \frac{2}{9} \qquad c = \frac{2}{9}$

$y = \frac{1}{9}(2e^{x^3} - 2 + x^3)$ v nule navázání na nulu (triviální řešení)

triviální řešení lze řešení

pro daní c až dva nulové body, další řešení napojit na $y=0$