(12) $(2x+1)y'' + 4xy' - 4y = 0$ dvě řešení

$x = -\frac{1}{2}$ $y = Ce^{-2x}$

$x < -\frac{1}{2}$

$y = e^{ax}$

$y' = ae^{ax}$

$y'' = a^2e^{ax}$

$(2x+1)a^2 + 4xa - 4 = 0$

$$a = \frac{-4 \pm \sqrt{4^2x^2 + 4^2(2x+1)}}{2(2x+1)} = 2\left(\frac{-x \pm \sqrt{(x+1)^2}}{2x+1}\right) =$$

$$= \begin{cases} \frac{2}{2x+1} \\ -2 \end{cases}$$

f.s. $\{e^{-2x}, u\}$

$x \neq -\frac{1}{2}$

$$W' = -\frac{4x}{2x+1}W$$

$$-\int_{x_0}^{x} \frac{4t}{2t+1}\,dt = -\int_{x_0}^{x} \frac{4t+2-2}{2t+1}\,dt = -\int_{x_0}^{x} 2 - \frac{2}{2x+1}$$

$$= \left[\ln|2t+1| - 2t\right]_{x_0}^{t}$$

$$W = W_0 e^{\left[\ln|2t+1| - 2t\right]_{x_0}^{x}} = \tilde{W}_0 (2x+1)e^{-2x}$$

$$\left(\frac{u}{e^{-2x}}\right)' = \int \frac{(2x+1)e^{-2x}}{e^{-4x}}\,dx = \int e^{2x}(2x+1)\,dx \quad \left| \begin{matrix} f = 2x+1 & f' = 2 \\ g' = e^{2x} & g = \frac{1}{2}e^{2x} \end{matrix} \right|$$

$$= \frac{1}{2}(2x+1)e^{2x} - \frac{2}{2}\int e^{2x} = \frac{1}{2}(2x+1)e^{2x} - \frac{1}{2}e^{2x} =$$

$$= xe^{2x}$$

$u = x$

f.s. $\{e^{-2x}, x\}$

$y_H = c_1e^{-2x} + c_2x$

(13) $xy'' + 2y - xy = 0 \qquad y = \frac{e^x}{x} \qquad x \neq 0$

f.s. $\left\{ \frac{e^x}{x}, u \right\}$ $\qquad x = 0 \quad y = 0$

$$W' = -\frac{a_{n-1}}{a_n} W(x) = -\frac{2}{x} W(x)$$

$$W = W_0 e^{-\int_{x_0}^{x} \frac{2}{t} dt} = W_0 e^{-[\ln|t|]_{x_0}^{x}}$$

volba $x_0 = 1$

$$W = W(x_0 = 1) \frac{1}{|x|}$$

nezáleží na multiplikativní konstantě

uvažujme $W = x$

$$\left(\frac{u}{\frac{e^x}{x}} \right)' = \frac{W}{\left(\frac{e^x}{x}\right)^2} = \frac{x^2}{x} \frac{1}{e^{2x}}$$

$$u = \frac{e^x}{x} \int \frac{x}{e^{2x}} = \frac{e^x}{x} \left[-2\frac{x}{e^{2x}} - \int e^{-2x} dx \right] =$$

$$= \frac{e^x}{x} \left(-2 \frac{x}{e^{2x}} + \frac{1}{4} e^{-2x} \right) = -\frac{1}{2e^x} + \frac{1}{4} \frac{1}{xe^{+x}} =$$

→ na konstantách (multiplikativních) nezáleží

$$\frac{(2x+1)}{e^{2x}} \frac{e^x}{x} = \frac{2}{e^x} + \frac{1}{xe^x}$$

$$\left\{ \frac{e^x}{x}, \frac{2}{e^x} + \frac{1}{xe^x} \right\} = \left\{ \frac{e^x}{x}, e^{-x}\left(2 + \frac{1}{x}\right) \right\}$$

$$\begin{vmatrix} \frac{e^x}{x} & (2x+1)\frac{e^{-x}}{x} \\ e^x\left(1 - \frac{1}{x^2}\right) & \left(-2e^{-x} + e^{-x}\left(-1 - \frac{1}{x^2}\right)\right) \end{vmatrix}$$

~~chceme~~ závislost f.s. $\left\{ \frac{e^x}{x}, e^{-x} \right\}$

$$\begin{vmatrix} \frac{e^x}{x} & e^{-x} \\ e^x\left(1 - \frac{1}{x^2}\right) & -e^{-x} \end{vmatrix} = -\frac{1}{x} - \left(1 - \frac{1}{x^2}\right) = -1 - \frac{1}{x} + \frac{1}{x^2}$$

musí být nulový! ale nám vadí jen vod

$$\begin{vmatrix} \frac{e^x}{x} & \frac{e^{-x}}{x} \\ e^x\left(1 - \frac{1}{x^2}\right) & e^x\left(-1 - \frac{1}{x^2}\right) \end{vmatrix} = -\frac{1}{x}\left(1 + \frac{1}{x^2}\right) - \frac{1}{x}\left(1 - \frac{1}{x^2}\right) =$$

$$-\frac{1}{x}\left(1 + \frac{1}{x^2} + 1 - \frac{1}{x^2}\right) = -\frac{2}{x}$$

nenulový ~~pro~~ $x \neq 0$

f.s $\left\{ \frac{e^x}{x}, \frac{e^{-x}}{x} \right\}$

(14) $(x+1)xy''+(x+2)y'-y = x+\frac{1}{x}$ $\quad y = x+2$

$\boxed{PROX \neq -1}$

$$\boxed{\begin{array}{c} x=-1 \\ y = Ce^{-t}-2 \end{array}}$$

f.s. $\{x+2, u\}$

$\to w' = -\frac{x+2}{x(x+1)} w(x)$

$-\int_{x_0}^{x} \frac{t+2}{t(t+1)} dt = -\int_{x_0}^{x} \frac{2}{t} - \frac{1}{t+1} dt = \left[\ln \frac{|t+1|}{t^2}\right]_{x_0}^{x}$

$w = w_0 \, e^{\left[\ln \frac{(t+1)}{t^2}\right]_{x_0}^{x}} = \tilde{w}_0 \frac{|x+1|}{x^2} = \tilde{w}_0 \frac{x+1}{x^2}$

$\tilde{w}_0 = 1$

$\left(\frac{u}{x+2}\right)' = \frac{w(x)}{(x+2)^2} = \frac{x+1}{x^2(x+2)^2}$

$\frac{u}{x+2} = \int \frac{x+1}{x^2(x+2)^2} dt = \int \frac{1}{4x^2} - \frac{1}{4(x+2)^2} dt =$

$= -\frac{1}{4}\left(\frac{1}{x} - \frac{1}{x+2}\right) = -\frac{1}{4} \frac{2}{x(x+2)} = -\frac{1}{2} \frac{1}{(x+2)\,x}$

$u = -\frac{1}{2}\frac{1}{x}$

$\Rightarrow$ f.s $\{x+2, \frac{1}{x}\}$

$y_H = c_1(x+2) + c_2\left(\frac{1}{x}\right)$

y_P: $\begin{pmatrix} x+2 & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{x^2+1}{x} \cdot \frac{1}{x} \cdot \frac{1}{x+1} \end{pmatrix}$

$c_1'(x+2) + \frac{1}{x} c_2' = 0 \qquad c_2' = -c_1' x(x+2)$

$c_1' \quad - \frac{c_2'}{x^2} = \frac{x^2+1}{x^2} \cdot \frac{1}{x+1}$

$c_1'\left(1 + \frac{x+2}{x}\right) = c_1'\left(\frac{x+x+2}{x}\right) = \frac{2(x+1)}{x} c_1' = \frac{x^2+1}{x^2} \frac{1}{x+1}$

$c_1' = \frac{1}{2}\left(\frac{x^2+1}{(x+1)^2} \frac{1}{x}\right) = \frac{1}{2}\left(\frac{1}{x} - \frac{2}{(x+1)^2}\right) \Rightarrow C_1 = \frac{1}{2}\left(\ln|x| + \frac{2}{x+1}\right)$

$c_2' = -x(x+2)c_1' = -\frac{1}{2}\frac{x^2+1}{(x+1)^2}(x+2) = -\frac{1}{2}\left(x + \frac{2}{(x+1)^2}\right) \Rightarrow$

$C_2 = -\frac{1}{2}\left(\frac{x^2}{2} - \frac{2}{x+1}\right)$

$$y_p = \frac{2+x}{x+1} + \frac{1}{2}(x+2)\ln|x| - \frac{x}{4} + \frac{1}{x(x+1)} =$$

$$= -\frac{x}{4} + \frac{1}{2}(x+2)\ln|x| + \frac{x^2+2x+1}{x(x+1)} =$$

$$= -\frac{x}{4} + \frac{1}{2}(x+2)\ln|x| + \frac{x+1}{x}$$

$$y = C_1(x+2) + \frac{C_2}{x} \left(-\frac{x}{4} + \frac{1}{2}(x+2)\ln|x| + \frac{x+1}{x} \right.$$

(15) $(2x+1)y'' + (2x-1)y' - 2y = x^2 + x$

jedno řešení polynomem $\boxed{\text{PRO } x \neq -\frac{1}{2}}$

$x = -\frac{1}{2}$

$y = Ce^{-x} + \frac{1}{8}$

→ $a_n x^n + \dots$

u y'' nejvyšší člen x^{n-1}

→ nejvyšší členy u y' a y musí se zrušit / stejné (ale musí se odečíst – hledáme homogenní řešení)

$$2x \cdot a_n n x^{n-1} - 2a_n x^n = 0$$

$$2x^n a_n (n-1) = 0$$

pro $n > 1$ $a_n = 0$ pro $n = 1$ $a_1 \in \mathbb{R}$

hledáme ve tvaru $y = ax + b$

$$(2x-1)a - 2(ax+b) = 0$$

$$x^0 \quad -a \quad -2b = 0$$

$$a = -2b$$

→ libovolná multiplikativní konstanta

$$y = ax + b = 2x - 1$$

f.s. $\{2x-1, u\}$

$$w' = -\frac{2x-1}{2x+1} w(x)$$

$$-\int_{x_0}^{x} \frac{2t-1}{2t+1}\,dt = -\int_{x_0}^{x} 1 - \frac{2}{2t+1}\,dt = +\left[\ln|2t+1| - t\right]_{x_0}^{x}$$

$$w = w_0 e^{+\left[\ln|2t+1| - t\right]_{x_0}^{x}} = \tilde{w}_0 e^{-x}|2x+1| = \tilde{w}_0 e^{-x}(2x+1)$$

$$\left(\frac{u}{2x-1}\right)' = \frac{w}{(2x-1)^2} = \frac{e^{-x}(2x+1)}{(2x-1)^2} \leftarrow \frac{u'v - uv'}{(2x-1)^2}, \quad v = 2x-1$$

$\frac{u}{2x-1} = -\frac{e^{-x}}{2x-1} \to u = -e^{-x}$, na multiplikativní konst. nezáleží

f.s. $\{2x-1, e^{-x}\}$

$$\begin{pmatrix} 2x-1 & e^{-x} \\ 2 & -e^{-x} \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{x^2+x}{2x+1} \end{pmatrix}$$

$$W = (-2x+1-2)e^{-x} = -(2x+1)e^{-x}$$

$$c_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ \frac{x^2+x}{2x+1} & -e^{-x} \end{vmatrix}}{W} = \frac{-e^{-x}\frac{x^2+x}{2x+1}}{-(2x+1)e^{-x}} = \frac{x^2+x}{(2x+1)^2} = \frac{1}{4} - \frac{1}{4(2x+1)^2}$$

$$c_2' = \frac{\begin{vmatrix} 2x-1 & 0 \\ 2 & \frac{x^2+x}{2x+1} \end{vmatrix}}{W} = -e^{x}\frac{(2x-1)(x^2+x)}{\underbrace{(2x+1)^2}}$$ uv+

$$c_1 = \frac{1}{4}\left(x + \frac{1}{2}\frac{1}{2x+1}\right) = \frac{1}{4}\,\frac{4x^2+2x+1}{2(2x+1)}$$

$$c_2 = -\frac{e^{x}(x^2-x-1)}{2x+1}$$

$$y_p = \underbrace{\frac{1}{8}\,\frac{4x^2+2x+1}{2x+1}(2x-1)} - \frac{x^2-x-1}{2x+1}$$

$$y = y_H + y_p = C_1(2x-1) + C_2 e^{x} + \frac{1}{8}\,\frac{(2x-1)(4x^2+2x+1) - 8x^2 - 8x - 1}{2x+1}$$

někde je num. chyba!

Jiné typy

- $y^{(n)} = f(x)$, $y^{(n)} = f(x)$, $y(x_0) = y_0, \dots, y^{(n-1)}(x_0) = y_{n-1}$

$$y = \int_{x_0}^{x} \int_{x_0}^{t_1} \dots \int_{x_0}^{t_{n-1}} f(t_n)\, dt_n \dots dt_1 + \sum_{k=0}^{n-1} \frac{y_k}{k!}(x-x_0)^k$$

- $y^{(n)} = f(y^{(n-1)}, x)$

$z(x) = y^{(n-1)}x \qquad z' = f(x, z)$

- $y^{(n)} = f(y^{(n-2)})$

$z'' = f(z) \qquad 2z'$

$(z')^2 = 2zz' = 2z' f(z)$

- $y^{(n)} = f(x, y^{k}, y^{(k+1)} \dots, y^{(n-1)})$

$z^{(n-k)} = f(x, z, z', \dots, z^{(n-k-1)})$

- $y^{(n)} = f(y, y', \dots, y^{(n-1)})$

$\rightarrow$ snížení o 1 řád

$y'(x) = h(y(x))$

$y''(x) = h'(y(x))y'(x) = h'(y(x))h(y(x))$

(16) $2yy' = y^2 + y'^2$ $\qquad (y^2)' = 2yy'$ $\qquad$ alespoň jedno řešení

$(y' - y)^2 = 0$

$y' - y = 0 \qquad \lambda - 1 = 0 \qquad e^x$

$y = c e^x$.

(17) $x^2 y'' = (y')^2 \qquad x = 0 \qquad y' = 0 \qquad y^{(n)} = f(x, y^{(n-1)})$

$x \neq 0 \quad y'' = \frac{y'^2}{x^2}$

$z = y'$

$z' = \frac{z^2}{x^2} \qquad z = 0$ triviální řešení

- $z \neq 0 \qquad I_1 = (0, \infty) \quad I_2 = (-\infty, 0)$
- $z \equiv 0$ triviální
- $z \quad J_1\ (0, \infty) \quad J_2\ (-\infty, 0)$
- $\frac{z'}{z^2} = \frac{1}{x^2}$

$-\frac{1}{z} = -\frac{1}{x} + \tilde{c}$

$\frac{1}{z} = \frac{1}{x} + c \qquad \frac{1}{x} \neq -c$

$z = \frac{1}{\frac{1}{x} + c} = \frac{x}{1 + xc}$

$y' = \frac{x}{1 + xc}$

$y = \int \frac{\cancel{x}}{1 + \cancel{x}c} \, dx = \frac{1}{c} \int \frac{c\cancel{x} + 1 - 1}{1 + \cancel{x}c} = \frac{1}{c} \int 1 - \frac{1}{1 + cx} \, dx$

$= \frac{1}{c} \left(x - \frac{1}{c} \ln|1 + cx| \right) + c_2$

$= \frac{x}{c} - \frac{1}{c^2} \ln|1 + cx| + c_2$

(18) $y''y^3 = 1$ $\qquad y \neq 0 \longleftarrow y'$ přinásobení (neekvivalentní řešení širší třída funkcí)

$(y'^2)' = 2y'y'' = 2y'\frac{1}{y^3}$

$y^{(n)} = f(y^{(n-2)})$

$y'^2 = -\frac{1}{y^2}$

$y' = \pm\sqrt{-\frac{1}{y^2}}$

(19) $y'' = e^y \qquad |\cdot y'$

$(y'^2)' = 2y''y' = 2y'e^y$

$(y'^2)' = 2y'e^y$

$y' = 2e^y + 2c$

$y' = \pm\sqrt{2e^y + 2c} \qquad -e^y > c$

$\frac{y'}{\pm\sqrt{2e^y+2c}} = 1$

$\pm\int \frac{dy}{\sqrt{2e^y+2c}} = x + c_2$

$\int \frac{dy}{\sqrt{2e^y+2c}} = \left| e^y = u \right| = \frac{1}{\sqrt{2}}\int \frac{1}{u\sqrt{u+c}}\,du =$

$\left| s = \sqrt{a+u} \right| = \frac{1}{\sqrt{2}}2\int \frac{1}{s^2-a}\,ds = -\frac{\sqrt{2}}{a}\int \underbrace{\frac{1}{1-\frac{s^2}{a}}}_{\text{argtanh}}\,ds$

(20) $y'' + (y')^2 = 2e^{-y}$ $\qquad$ $y^{(n)} = f(y, y', \dots, y^{n-1})$

$y'(x) = p(y(x)) = p(z)$ $\qquad$ $y = z$

$y''(x) = p'(y(x))\, p(y(x))$

$p'(z)\, p(z) + p^2(z) = 2e^{-z}$ $\qquad$ $p(z) = 0$

$p(z) + p^2(z) = \frac{2}{p(z)} e^{-z}$ $\qquad$ Bernoulli

$\alpha = -1$

$\cancel{g(z)}\ p^{1+1} = \eta^2(z)$

$\eta' = 2pp' \to$

$\frac{1}{2}\eta' + \eta = 2e^{-z}$

$\eta' + 2\eta = 4e^{-z}$

$\to \eta_H$ $\quad$ $\eta' = -2\eta$ $\qquad$ $\eta \neq 0$

$\frac{\eta'}{\eta} = -2$ $\qquad$ $\ln|\eta| = -2z + c$

$\eta = c e^{-2z}$

$\eta_p: \ C(z) e^{-2z}$

$\eta_p' = C'(z) e^{-2z} + C(z)(-2) e^{-2z}$

$C'(z) e^{-2z} + C(z)(-2) e^{-2z} + 2C(z) e^{-2z} = 4e^{-z}$

$C'(z) = 4e^{z}$

$C(z) = 4e^{z}$

$\eta_p = 4e^{-z}$

$\eta = Ce^{-2z} + 4e^{-z}$

$\eta = \cancel{p}^2 \to \cancel{p} = \pm\sqrt{Ce^{-2z} + 4e^{-z}} = e^{-z}\sqrt{C + 4e^{z}}$

$y' = p = \pm\sqrt{Ce^{-2z} + 4e^{-z}} = \pm\sqrt{Ce^{-2y} + 4e^{-y}}$ neprinosid

$1 = \frac{y'}{\pm\sqrt{Ce^{-2y} + 4e^{-y}}} = \frac{y'}{e^{-y}\sqrt{C + 4e^{y}}}$ $\quad y$

pro $C \neq 4e^y$ $\rightarrow$

$$\int \frac{dy}{e^{-y}\sqrt{C+4e^y}} = \left| \begin{array}{l} u = e^y \\ \frac{du}{dy} = e^y \end{array} \right| = \int \frac{du}{\sqrt{C+4u}} =$$

$$= \frac{2}{4}\sqrt{C+4u} = x + C_2 \qquad x + C_2 > 0$$

$$x > C_2$$

$$C + 4u = \Big(\overbrace{2(x+C_2)}^{>0}\Big)^2$$

$$e^y = u = (x+C_2)^2 - \frac{C}{4} \qquad (C_2+x)^2 - \frac{C}{4} > 0$$

$$y = \ln\left((x+C_2)^2 - \frac{C}{4}\right)$$