① $y''' - 3y'' + 3y' - y = 0$

$e^{\lambda x}$

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0$

$(\lambda - 1)^3 = 0$ → trojnásobný kořen. $\lambda = 1$

$\{e^x, xe^x, x^2e^x\}$ fundamentální systém

$y = C_1e^x + C_2xe^x + C_3x^2e^x \quad C_1, C_2, C_3 \in R$

② $y'' - 2y' - 3y = e^{4x}$

→ homogenní rovnice

→ char. polynom $\lambda^2 - 2\lambda - 3 = 0$

$(\lambda + 1)(\lambda - 3) = 0$

→ fund. systém $\{e^{-x}, e^{3x}\}$

→ $\underline{y_H = C_1e^{-x} + C_2e^{3x}}$

→ partikulární řešení

metoda neurčitých koeficientů, speciální pravá strana

e^{4x} $\quad \lambda = 4$ není řešením f.s.

$f(x) = e^{\mu x}(P_1(x)\cos\nu x + P_2 \sin\nu x)$

$y_p = e^{\mu x}x^k(Q_1(x)\cos\nu x + Q_2(x)\sin\nu(x))$

$k = 0 \quad \cancel{BL}\ P_1 = 1 \quad P_2 = 0 \quad \sin 0\cdot x = 0$

y_p hledáme ve tvaru $y_p = Ae^{4x}$

$y_p' = 4Ae^{4x}$

$y_p'' = 4^2Ae^{4x}$

dosazení

$4^2Ae^{4x} - 2\cdot 4Ae^{4x} - 3Ae^{4x} = e^{4x}$

$5A = 1 \to A = \frac{1}{5}$

$y_p = \frac{e^{4x}}{5}$

→ variace konstant

$$\begin{pmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{4x} \end{pmatrix}$$

$e^{-x}C_1' + e^{3x}C_2' = 0$

$-e^{-x}C_1' + 3e^{3x}C_2' = e^{4x}$

$C_1' = -e^{4x}C_2'$

$4e^{3x}C_2' = e^{4x}$

$C_2' = \frac{1}{4}e^x$

$C_2 = \frac{1}{4}e^x$

$C_1' = -\frac{1}{4}e^{5x}$

$C_1 = -\frac{1}{4\cdot 5}e^{5x}$

$y_p = -\frac{1}{4\cdot 5}e^{4x} + \frac{1}{4}e^{4x} = \frac{e^{4x}}{5}$

$y = C_1e^{-x} + C_2e^{3x} + \frac{e^{4x}}{5}$

③ $y'' - y = 2e^x - x^2$

- Char. rovnice $\lambda^2 - 1 = 0 \quad \lambda = \pm 1$
- $\{e^{-x}, e^{x}\}$
- $y_H = C_1 e^{-x} + C_2 e^{x}$
- $y_{p1} + y_{p2}$

$y_{p1} \Leftrightarrow 2e^x \quad \lambda = +1$ – 1 násobné, st $P = 0$

$y_{p2} \Leftrightarrow -x^2 \quad \lambda = 0$ 0 násobné st $P = 2$

$y_{p1} = Ae^{x} \cdot x$

$$\left.\begin{aligned} y_{p1}' &= e^{x}\cdot x + e^{x} = e^{x}(x+1) \\ y_{p1}'' &= e^{x}(x+1) + e^{x} = e^{x}(x+2) \\ Ae^{x}(x+2) &- Ae^{x}\cdot x = 2e^{x} \\ 2A &= 2 \\ A &= 1 \end{aligned}\right\} \Rightarrow y_{p1} = xe^{x}$$

$$\left.\begin{aligned} y_{p2} &= Bx^2 + Cx + D \\ y_{p2}' &= 2Bx + C \\ y_{p2}'' &= 2B \\ 2B &- Bx^2 + Cx + D = -x^2 \\ x^2&:\ B = 1 \\ x^1&:\ C = 0 \\ x^0&:\ 2 - D = 0 \to D = 2 \end{aligned}\right\} y_{p2} = x^2 + 2$$

$y = y_H + y_{p1} + y_{p2} = C_1 e^{-x} + C_2 e^{x} + xe^{x} + x^2 + 2$

(4) $y'' - 3y' + 2y = \sin x$

- char. polynom

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

f.s. $\{e^x, e^{2x}\}$

- $y_H = C_1 e^x + C_2 e^{2x}$

- y_p: $\sin x \to \lambda + i\gamma = 0 + i$

$st. P_2 = 0$

$\to y_p = e^{0 \cdot x} \cdot x^0 \cdot (A\cos x + B\sin x) = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$y_p'' - 3y_p' + 2y_p = \sin x$

$-A\cos x - B\sin x + 3A\sin x - 3B\cos x + 2A\cos x + 2B\sin x = \sin x$

$\sin x$: $-B + 3A + 2B = 1$

$3A + B = 1$

$\cos x$: $-A - 3B + 2A = 0$

$-3B + A = 0$

$A = 3B$

$9B + B = 1 \to B = \frac{1}{10}$

$A = \frac{3}{10}$

$y_p = \frac{3}{10}\cos x + \frac{1}{10}\sin x$

$y = y_H + y_p = C_1 e^x + C_2 e^{2x} + \frac{3}{10}\cos x + \frac{1}{10}\sin x$

⑤ $y'' + 4y' - 5y = 2e^x \sin^2 x$

• char rovnice

$\lambda^2 + 4\lambda - 5 = 0 \longrightarrow (\lambda + 5)(\lambda - 1) = 0$

• f.s. $\{e^x, e^{-5x}\}$

homogenní řešení

$\boxed{y_H = C_1 e^x + C_2 e^{-5x}}$

• y_p: metoda variace konstant, nebo po úpravě speciální pravá strana a metoda neurčitých koeficientů

metoda variace konstant, nebo po úpravě speciální pravá strana a metoda neurčitých koeficientů

identita: cos^2x-sin^2x=cos(2x)

$\sin^2 x = 1 - \cos^2 x = 1 - (\cos^2 x - \sin^2 x) - \sin^2 x = 1 - \sin^2 x - \cos 2x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

takto jsme upravili pravou stranu

chtěli jsme se zbavit toho sin^2(x)

$f(x) = \boxed{e^x - e^x \cos 2x}$

násobnost kořenu

• y_{p1} $\quad f_1(x) = e^x$ násobnost kořenu $\lambda = 1$ je 1

to krát x tam je proto, že řešení e^x už jednou je v homogenním řešení

$y_{p1} = A e^x x$

$y_{p1}' = A e^x (1 + x)$

$y_{p1}'' = A e^x (2 + x)$

$A(2+x) + 4(1+x) - 5Ax = 1$

$x^1: \quad A + 4A - 5A = 0$ ✓

$x^0: \quad 2A + 4A = 1$

$A = \frac{1}{6}$

první partikulární řešení k tomu e^x

$\boxed{y_{p1} = \frac{x e^x}{6}}$

• y_{p2} $\quad f_2(x) = -e^x \cos 2x$ $\quad \lambda = 1 + 2i$, násobnost

$k, p = 0$

$y_p = e^x (A \cos 2x + B \sin 2x)$

$y_p' = e^x (A \cos 2x + B \sin 2x - 2A \sin 2x + 2B \cos 2x)$

$y_p'' = e^x (A \cos 2x + B \sin 2x - 2A \sin 2x + 2B \cos 2x$

$\quad - 2A \sin 2x + 2B \cos 2x - 4A \cancel{\cos} \cos 2x - 4B \sin 2x)$

dosadili jsme ty derivace yp do té levé strany

$A \sin 2x \, (-2 - 4 - 8) = 0$

$B \sin 2x \, (1 - 4 + 4 - 5)$ $\quad 0$ na pravé straně není žádné sin2x

$A \cos 2x \, (1 - 4 + 4 - 5)$

$B \cos 2x \, (2 + 2 + 8)$ $\quad \} -1$ protože na pravé straně je -cos2x

$-10A - 4B = 0$

$-4A + 12B = -1$

$12B = -36A$

$A = \frac{1}{40}$

$B = -\frac{3}{40}$

druhé partikulární řešení k tomu e^xcos(2x)

$\boxed{y_{p2} = e^x \left(\frac{1}{40} \cos 2x - \frac{3}{40} \sin 2x \right)}$

Výsledek:

$y = y_H + y_{p1} + y_{p2} = \underbrace{C_1 e^x + C_2 e^{-5x}}_{\text{homogenní řešení}} + \underbrace{\frac{x e^x}{6}}_{\text{první partikulární}} + \underbrace{e^x \left(\frac{1}{40} \cos 2x - \frac{3}{40} \sin 2x \right)}_{\text{druhé partikulární}}$

⑥ $y'' - 2y' + y = 2xe^x + e^x \sin 2x$

• char. rce $\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$

• F.S. $\{e^x, xe^x\}$

• $y_H = C_1 e^x + C_2 x e^x$

• $f(x) = 2xe^x + e^x \sin 2x$

$\underbrace{\lambda = 1}_{\text{2násobný}}$ $\quad$ $\lambda = 1 + 2i$

0-násobný

$kp = 1$ $\quad$ $kp = 0$

• y_{p1} $\quad f_1(x) = 2xe^x$

Jednotlivé derivace

$y_{p1} = (Ax + B)x^2 e^x = (Ax^3 + Bx^2)e^x$

$y_{p1}' = (Ax^3 + Bx^2 + 3Ax^2 + 2Bx)e^x = (Ax^3 + (B+3A)x^2 + 2Bx)e^x$

$y_{p1}'' = (Ax^3 + (B+3A)x^2 + 2Bx + 3Ax^2 + 2(B+3A)x + 2B)e^x =$

$(Ax^3 + (B+6A)x^2 + (4B+6A)x + 2B)e^x$ $\quad = 0$

Dosazení

$y_{p1}'' - 2y_{p1}' + y_{p1} = 2xe^x$

x^3: $A - 2A + A = 0 = 0$ ✓

x^2: $B + 6A - 2(B + 3A) + B = 0$

x: $4B + 6A - 4(B \quad) = 2 \Rightarrow A = \frac{1}{3}$

$B = 0$

x^0:

$\boxed{y_{p1} = \frac{x^3 e^x}{3}}$

y_{p2} $\quad f_2 = e^x \sin 2x$

$y_{p2} = e^x(\underline{A\cos 2x} + B\sin 2x)$

$y_{p2}' = e^x(\underline{A\cos 2x} + B\sin 2x - 2A\sin 2x + \underline{2B\cos 2x})$

$y_{p2}'' = e^x(\underline{A\cos 2x} + B\sin 2x - 2A\sin 2x + \underline{2B\cos 2x} - \underline{4A\cos 2x} - 4B\sin 2x$

$-2A\sin 2x + \underline{2B\cos 2x})$

$\cos 2x$: $\cancel{A + 2B + 2B} - 4A - 2A - 4B + A = 0 \Rightarrow A = 0$

$\sin 2x$: $B - 2A - 2A - 4B - 2B + 4A + B = 1 \Rightarrow B = -\frac{1}{4}$

$\boxed{y_{p2} = -\frac{1}{4} e^x \cdot \sin 2x}$

$\boxed{y = C_1 e^x + C_2 x e^x + \frac{x^3 e^x}{3} - \frac{1}{4} e^x \sin 2x}$

⑦ $y^{IV} - 5y'' + 4y = \sin x \cos 2x$

· char. rac. $\lambda^4 - 5\lambda^2 + 4 = 0$

$\mu = \lambda^2$

$\mu^2 - 5\mu + 4 = 0$

$(\mu - 4)(\mu - 1) = 0$

$\mu_1 = 4 \to \lambda_1 = 2,\ \lambda_2 = -2$

$\mu_2 = 1 \to \lambda_3 = +1,\ \lambda_4 = -1$

· f.s. $\{e^{x}, e^{-x}, e^{+2x}, e^{-2x}\}$

· $y_H = C_1 e^{x} + C_2 e^{-x} + C_3 e^{2x} + C_4 e^{-2x}$

what??

• $\sin x \cos 2x = \frac{1}{2i}(e^{ix} - e^{-ix}) \cdot \frac{1}{2}(e^{2ix} + e^{-2ix}) =$

$= \frac{1}{2} \cdot \frac{1}{2i}(e^{3ix} + e^{-ix} - e^{ix} - e^{-3ix}) = \frac{1}{2}(\sin 3x - \sin x)$

$f(x) = \sin x \cos 2x = \underbrace{\frac{1}{2}(\sin 3x}_{\substack{\lambda = 0+3i \\ \text{st } P = 0 \\ \text{nás } 0}} - \underbrace{\sin x)}_{\substack{\lambda = 0+i \\ \text{st } P = 0 \\ \text{nás } 0}}$

• $y_{p1} \leftarrow f_1(x) = \frac{1}{2}\sin 3x$

$y_{p1} = A \cos 3x + B \sin 3x$

$y_{p1}' = -3A \sin 3x + B \cdot 3 \cos 3x$

$y_{p1}'' = -3^2 A \cos 3x - 3^2 B \sin 3x$

$y_{p1}''' = 3^3 A \sin 3x - 3^3 B \cos 3x$

$y_{p1}^{IV} = 3^4 A \cos x + 3^4 B \sin x$

$\cos 3x$: $3^4 A - 3^2 A(-5) + 4A = 0$

$A = 0$

$\sin 3x$: $3^4 B - 3^2 B(-5) + 4B = \frac{1}{2}$

$3^2(9+5) + 4$

$(\underbrace{9 \cdot 14} + 4)B = \frac{1}{2}$

$126 + 4 \quad B = \frac{1}{2}$

$B = \frac{1}{260}$

· $y_{p2} \leftarrow f_2(x) = -\frac{1}{2}\sin x$

$y_{p2} = A \cos x + B \sin x$

$y_{p2}' = -A \sin x + B \cos x$

$y_{p2}'' = -A \cos x - B \sin x$

$y_{p2}''' = A \sin x - B \cos x$

$y_{p2}^{IV} = A \cos x + B \sin x$

$\to \quad A = 0$

B: $B + 5B + 4B = -\frac{1}{2}$

$B = -\frac{1}{20}$

$$\boxed{y = C_1 e^{x} + C_2 e^{-x} + C_3 e^{2x} + C_4 e^{-2x} + \frac{\sin 3x}{260} - \frac{\sin x}{20}}$$

(8) $y'' - 2y' + y = \frac{e^x}{x}$ $x \neq 0$

- char. rce.

$\lambda^2 - 2\lambda + 1 = 0$

$(\lambda - 1)^2 = 0$ $\lambda_{1,2} = 1$

- f.s. $\{e^x, xe^x\}$

homogenní řešení

$$\boxed{y_H = C_1 e^x + C_2 x e^x}$$

- y_p metoda variace konstant

metoda variace konstant

Nezderivované členy F.S.

Zderivované členy F.S.

$$\begin{pmatrix} e^x & xe^x \\ e^x & (1+x)e^x \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^x}{x} \\ \hline 1 \end{pmatrix}$$

ten člen na pravé straně

Máme SLR, vyřešíme

$c_1' e^x + x e^x c_2' = 0$

$c_1' e^x + (1+x) e^x c_2' = \frac{e^x}{x}$

odečtení (2)−(1)

odečteme od druhé rovnice první

Nalezneme C1 a C2

$e^x c_2' = \frac{e^x}{x} \to c_2' = \frac{1}{x} \to c_2 = \ln|x|$

$c_1' = -1 \to c_1 = -x$

Partikulární řešení

$$\boxed{y_p = \ln|x| \, x e^x - x e^x} = e^x x(\ln|x| - 1)$$

$y = y_H + y_p = C_1 e^x + C_2 x e^x + e^x x \ln|x| - e^x x =$

$= C_1 e^x + (C_2 - 1) x e^x + e^x x \ln|x| = C_1 e^x + \tilde{C}_2 x e^x + e^x x \ln|x|$

Výsledek

⑨ $y'' + 4y = 2\,\mathrm{tg}\,x \qquad x \neq \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$

- char. rce $\lambda^2 + 2 = 0$

  $\lambda = \pm 2i$

  f.s. $\{\sin 2x, \cos 2x\}$ f.s $\{e^{2i}, e^{-2i}\}$

- $y_H = C_1 \cos 2x + C_2 \sin 2x$

- y_p variace konstant

$$\begin{pmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{pmatrix} \begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 2\,\mathrm{tg}\,x \end{pmatrix}$$

Wronskián

Crammerovo pravidlo $\det W = 2\cos^2 2x + 2\sin^2 2x = 2$

$$\begin{vmatrix} 0 & \sin 2x \\ 2\,\mathrm{tg}\,x & 2\cos 2x \end{vmatrix} = -2\sin 2x\,\mathrm{tg}\,x \qquad \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & 2\,\mathrm{tg}\,x \end{vmatrix} = 2\cos 2x\,\mathrm{tg}\,x$$

$$C_1' = \frac{-2\sin 2x\,\mathrm{tg}\,x}{2} = -2\sin x \cos x\,\mathrm{tg}\,x = -2\sin^2 x = -(1 - \cos 2x)$$

$$C_1 = -\left(x - \frac{\sin 2x}{2}\right)$$

$$C_2' = \cos 2x\,\mathrm{tg}\,x = (\cos^2 x - \sin^2 x)\frac{\sin x}{\cos x} = (2\cos^2 x - 1)\frac{\sin x}{\cos x} =$$

$$= 2\sin x \cos x - \frac{\sin x}{\cos x} = \sin 2x - \frac{\sin x}{\cos x}$$

$$C_2 = -\frac{\cos 2x}{2} + \ln|\cos x|$$

$$y_p = -\left(x - \frac{\sin 2x}{2}\right)\cos 2x + \sin 2x\left(-\frac{\cos 2x}{2} + \ln|\cos x|\right) =$$

$$= \cos 2x\left(-x + \frac{\sin 2x}{2} - \frac{\sin 2x}{2}\right) + \sin 2x \ln|\cos x|$$

$$= -x\cos 2x + \sin 2x \ln|\cos x|$$

- $y = C_1 \cos 2x + C_2 \sin 2x - x\cos 2x + \sin 2x \ln|\cos x|$

(10) $x^2 y''' = 2y'$ $\quad x \neq 0$ $\quad$ Eulerova rovnice

Eulerova rovnice

$x^3 y''' - 2xy' = 0$

→ hledáme řešení ve tvaru x^λ

$(x^\lambda)' = \lambda x^{\lambda-1}$, $(x^\lambda)'' = \lambda(\lambda-1)x^{\lambda-2}$, $(x^\lambda)''' = \lambda(\lambda-1)(\lambda-2)x^{\lambda-3}$

$\big(\lambda(\lambda-1)(\lambda-2) - 2\lambda\big)x^\lambda = 0 \qquad x \neq 0$

$\lambda\big((\lambda-1)(\lambda-2) - 2\big) = \lambda(\lambda^2 - 3\lambda) = \lambda^2(\lambda-3) = 0$

$\lambda_{1,2} = 0, \quad \lambda_3 = 3$

• f.s. $\{1, \ln|x|, x^3\}$

• $y = c_1 + c_2 \ln|x| + c_3 x^3$

• substituce

$\xi = \ln x \to x = e^\xi$

$z(\xi) := y(x(\xi)) = y(e^\xi) =$

$z'(\xi) = \frac{dy}{d\xi} = \frac{dy}{dx}\frac{dx}{d\xi} = y'e^\xi = y'x$

$z''(\xi) = y''x^2 + y'(e^\xi)' = y''x^2 + y'x$

$z'''(\xi) = y'''x^3 + \underbrace{y''x^2}_{z''} + 2y''x^2 + \underbrace{y'x}$

$\qquad = 2z''(\xi) - 2y'x = 2z''(\xi) - 2z'(\xi)$

$z'''(\xi) - z'' + 2z' - 2z'' - 2z' = 0$

$z'''(\xi) - 3z'' = 0$

char. r. $\lambda^3 - 3\lambda^2 = \lambda^2(\lambda - 3)$

f.s. $\{\xi e^{0\cdot\xi}, e^{0\cdot\xi}, e^{3\xi}\}$

v x $\{\ln|x|, 1, x^3\}$

(11) $x^2y'' + xy' + 4y = 10x$

$\to x^\lambda$

$\lambda(\lambda-1) + \lambda + 4 = 0$

$\lambda^2 - \lambda + \lambda + 4 = 0$

$\lambda^2 = -4$

$\lambda = \pm 2i$

- f.s $\{\cos 2\ln|x|, \sin 2\ln|x|\}$

$\to \xi = \ln|x| \qquad x = e^\xi$

$z(\xi) = y(x(\xi)) \qquad z'(\xi) = y'x$

$z''(\xi) = y''x^2 + y'x$

$z'' + 4z = 10e^\xi$

$\lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$

f.s $\{\cos 2\xi, \sin 2\xi\}$

$y_H = c_1\cos 2\xi + c_2 \sin 2\xi$

$z_p \quad Ae^\xi$

$z_p' = Ae^\xi$

$z_p'' = Ae^\xi$

$5A = 10 \to A = 2$

$z_p = 2e^\xi$

$z = c_1 \cos 2\xi + c_2 \sin 2\xi + 2e^\xi$

$y = c_1 \cos 2\ln|x| + c_2 \sin 2\ln|x| + 2x$

$e^{\alpha\xi}\,\xi^k (P_1(\xi)\cos\beta\xi + P_2(\xi)\sin\beta\xi)$

$\xi \to \ln|x|$

$\xi(\ln x)^k$

$[P_1(\ln x)\cos\beta\ln x$

$P_2(\ln x)\sin\beta\ln x$

$y_p = Ax$

$y_p' = A$

$y_p'' = 0$

$5A = 10 \to A = 2$

$y = c_1\cos 2\ln|x| + c_2 \sin 2\ln|x| + 2x$