• $f, g: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}^*$

• $f = o(g)$ pro $x \to x_0$, jestliže $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$

• $f = O(g)$ pro $x \to x_0$, jestliže $\exists K, \delta > 0$ takové, že $|f(x)| \leq K|g(x)|$ na $P_\delta(x_0)$

• f je slabě ekvivalentní s g pro $x \to x_0$, jestliže $\lim_{x \to x_0} \frac{f(x)}{g(x)} \in \mathbb{R} \setminus \{0\}$ $\quad f \sim g$

• f je silně ekvivalentní s g pro $x \to x_0$, jestliže $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$, $f \simeq g$

$f = o(g) \Rightarrow f = O(g)$

$f \simeq g \Rightarrow f \sim g \Rightarrow f = O(g) \wedge g = O(f)$

• $\lim_{x \to x_0} \frac{f(x)}{g(x)} = +\infty$ značíme jako $g \ll f$ pro $x \to x_0$

• $x \to \infty$ $\quad 0 < \alpha < \beta$

$e^{-x} \ll x^{-\beta} \ll x^{-\alpha} \ll \frac{1}{\ln x} \ll 1 \ll \ln x \ll x^{\alpha} \ll x^{\beta} \ll e^{x}$

$\lim_{x \to \infty} \frac{\ln x}{x^\alpha} = 0 \quad \forall \alpha > 0$

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\operatorname{tg} x}{x} = \lim_{x \to 0} \frac{\ln(x+1)}{x} = \lim_{x \to 0} \frac{e^x - 1}{x} = 1$

$x = \frac{a}{n^\alpha} \quad \alpha > 0 \quad x \to 0 \Rightarrow n \to \infty$

• $x \to 0^+$ $\quad 0 < \alpha < \beta$

$x^{\beta} \ll x^{\alpha} \ll \frac{1}{\ln(\frac{1}{x})} \ll 1 \ll \ln \frac{1}{x} \ll x^{-\alpha} \ll x^{-\beta}$

• Stirling & faktoriál & Eulerovo číslo

$\left(\frac{n}{e}\right)^n < \frac{n!}{e} < n\left(\frac{n}{e}\right)^n$, $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

① n-tý částečný součet $\sum_{n=1}^{\infty} n^2$

	1	2	3	4	5
$S(n)$	1	3	6	10	15
$S(n^2)$	1	5	14	30	55
n^2	1	4	9	16	25

$$S_n = 1 + 2 + 3 + \dots + n$$
$$S_n = n + n-1 + n-2 + \dots + 1$$
$$2S_n = \underbrace{(n+1) + (n+1) + \dots + (n+1)}_{n\times} \to S_n = \frac{n}{2}(n+1)$$

- $k^3 - (k-1)^3 = k^3 - (k^3 - 3k^2 + 3k - 1) = 3k^2 - 3k + 1 \quad \sum_{k=1}^{n}$

$$\underbrace{\sum_{k=1}^{n} k^3 - (k-1)^3} = \sum_{k=1}^{n} 3k^2 - \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

$k=1$: $1 - 0$

$k=2$: $2 - 1$

$k=3$: $3 - 2$

$\vdots$

$k=n-1$: $(n-1)^3 - (n-2)^3$

$k=n$: $n^3 - (n-1)^3$

$$n^3 = 3\sum_{k=1}^{n} k^2 - 3\frac{n}{2}(n+1) + n$$

$$3\sum_{k=1}^{n} k^2 = n^3 + \frac{3}{2}n(n+1) - n =$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}\left(2n^3 + 3n^2 + 3n - 2n\right) = \frac{1}{6}n(2n^2 + 3n + 1) =$$
$$= \frac{1}{6}n(2n+1)(n+1)$$

② Vypočítej $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{[\frac{n}{2}]}}{2^n}$ $\quad -1^{[\frac{n}{2}]}$

1	2	3	4	5
1	-1	-1	1	1

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\frac{n}{2}]}}{2^n} = \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} - \dots\dots$$

$$= \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2^{4k}} + \sum_{k=1}^{\infty} \frac{1}{2^{4k+1}} - \sum_{k=1}^{\infty} \frac{1}{2^{4k-2}} - \sum_{k=1}^{\infty} \frac{1}{2^{4k-1}}$$

konverguje i absolutně!!
součty permutuji!

$$\sum_{k=1}^{\infty} \frac{1}{2^{4k}} = \lim_{n\to\infty} S_n \qquad \underline{\text{geometrická řada}}$$

Součet geometrické řady

$$S = a + ar + ar^2 + \dots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k$$

$$rS = ra + ar^2 + ar^2 + \dots + ar^n = \sum_{k=1}^{n} ar^k$$

$$S - rS = a - ar^n$$ sčítám od $\underline{0 \text{ do } n-1}$!

$$S(1-r) = a(1-r^n) \Rightarrow S = a\,\frac{1-r^n}{1-r}$$

$$\sum_{k=0}^{n} \frac{1}{2^{4k}} = 1 \cdot \frac{1-\left(\frac{1}{2}\right)^{4(n+1)}}{1-\frac{1}{16}} = \frac{1-\left(\frac{1}{2}\right)^{4(n+1)}}{\frac{15}{16}} \xrightarrow{n\to\infty} \frac{16}{15}$$

$$\sum_{k=1}^{n} \frac{1}{2^{4k}} = \sum_{k=0}^{n} \frac{1}{2^{4k}} - 1 = \frac{1}{15}$$

suma celkem $\displaystyle \frac{1}{2} + \frac{1}{15} + \frac{1}{2}\frac{1}{15} - 4\frac{1}{15} - 2\frac{1}{15} = \frac{1}{2} + \frac{\overbrace{2+1-8-4}^{-9}}{30}$

$$= \frac{15-9}{30} = \frac{6}{30} = \frac{1}{5}$$

③ $\sum_{n=1}^{\infty}(a+nd)q^n \qquad a,d\in\mathbb{R} \qquad |q|<1$

$$\sum_{n=1}^{\infty} aq^n + \sum_{n=1}^{\infty} ndq^n$$

- $\sum_{n=1}^{\infty} aq^n = q\sum_{n=1}^{\infty} aq^{n-1} = q\sum_{n=0}^{\infty} aq^n =$

$$= \lim_{n\to\infty} qa\,\frac{1-q^n}{1-q} \xrightarrow{|q|<1} \frac{qa}{1-q}$$

- $\sum_{n=1}^{\infty} ndq^n = \lim_{n\to\infty} d(q+2q^2+\dots+nq^n)$

$$nq^n: \left.\begin{matrix} q+q^2+\dots\dots+q^n \\ q^2+\dots\dots+q^n \\ q^3+\dots q^n \\ \vdots \\ q^n \end{matrix}\right\} n \qquad \begin{matrix} q\,\frac{1-q^n}{1-q} \to \frac{q}{1-q} \\ q^2\,\frac{1-q^{n-1}}{1-q} \to \frac{q^2}{1-q} \\ \vdots \\ q^n\,\frac{1-q}{1-q} \to \end{matrix}$$

$$\frac{q}{1-q}+\frac{q^2}{1-q}+\frac{q^3}{1-q}\dots\dots = \frac{q}{1-q}(\underbrace{1+q+q^2+\dots\dots}_{n\text{-krát}}) =$$

$$\frac{q}{1-q}\,\frac{1-q^n}{1-q} \to \frac{q}{(1-q)^2}$$

$$\sum_{n=1}^{\infty}(a+nd)q^n = \frac{qa}{1-q}+\frac{dq}{(1-q)^2} = \frac{q(1-q)a+dq}{(1-q)^2} =$$

$$= \frac{q(a+d)-aq^2}{(1-q)^2}$$

Pozn $\sum_{k=1}^{\infty}\sum_{j=1}^{k} a_{k+1-j}\,b_j$

④ $\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$

n	1	2	3	4	5
$\frac{1}{n(n+3)}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{18}$	$\frac{1}{28}$	$\frac{1}{40}$

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$1 = A(\ \ +3) + B\cdot$

$n=0 \quad A = \frac{1}{3}$

$n=-3 \quad B = -\frac{1}{3}$

$$\frac{1}{n(n+3)} = \frac{1}{3}\left(\frac{1}{n} - \frac{1}{n+3}\right)$$

$$\lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} \ldots\right.$$

$$\left. - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}\right) = \lim_{n\to\infty} \frac{1}{3}\Big(1 + \frac{1}{2} + \frac{1}{3} - \underbrace{\frac{1}{n+1}}_{\to 0} - \underbrace{\frac{1}{n+2}}_{\to 0} + \underbrace{\frac{1}{n+3}}_{\to 0}\Big) =$$

$$= \frac{1}{3} \cdot \frac{11}{6} = \frac{11}{18}$$

TELESKOPICKÁ ŘADA

⑤ $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

$$\frac{2n+1}{n^2(n+1)^2} = \frac{A}{n+1} + \frac{B}{(n+1)^2} + \frac{C}{n} + \frac{D}{n^2}$$

$$2n+1 = A(n+1)n^2 + Bn^2 + Cn(n+1)^2 + D(n+1)^2$$

$n=0 \quad 1 = D \qquad n^3: \quad 0 = A + C$

$n=-1 \quad -1 = B \qquad n^2: \quad 0 = A + B + 2C + D \quad \Big\}\ A = C = 0$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+1)^2} \quad n+1 = n'$$

$$= 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{9} + \frac{1}{3} - \frac{1}{16} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n'=2}^{\infty} \frac{1}{n'^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} - \frac{1}{n^2} = 1$$

(6) • $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$0 \le \frac{1}{n^2+1} \le \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$

$0 \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le \sum_{n=1}^{\infty} \frac{1}{n-1} - \frac{1}{n} = 1 \rightarrow$ Konverguje

• $\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)} = \underbrace{\overbrace{\sum_{n=1}^{\infty} \frac{1}{n+2}}}_{A} + \underbrace{\overbrace{\sum_{n=1}^{\infty} \frac{1}{n(n+2)}}}_{B}$

B: konverguje (viz předchozí)

A: $\sum_{n=1}^{\infty} \frac{1}{n+2} = \sum_{n=1}^{\infty} \frac{1}{n} - 1 - \frac{1}{2}$

? konverguje $\sum \frac{1}{n}$ harmonická řada?

$s_1 = 1 \quad s_2 = 1 + \frac{1}{2} = \frac{3}{2} \quad s_3 = \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2} = 2$

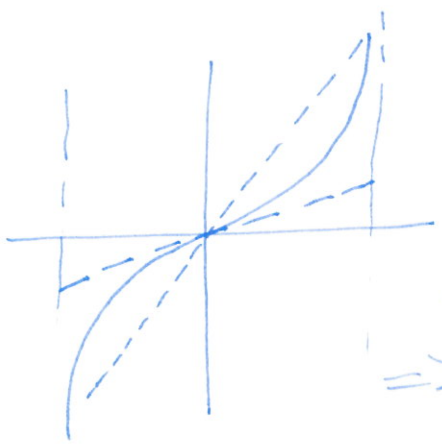
$s_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{8} > 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{1/2} = \frac{5}{2}$

$s_{2^n} > \frac{n+2}{2} \xrightarrow{n\to\infty} \infty$ a tedy A nekonverguje

$\Rightarrow \sum \frac{n+1}{n(n+2)}$ nekonverguje

• $\sum_{n=1}^{\infty} \operatorname{tg} \frac{\pi}{4n}$ pro dostatečně velká n $\operatorname{tg}\frac{\pi}{4n} \sim \frac{\pi}{4n}$

$\sum \frac{\pi}{4n}$ nekonverguje



pro dostatečně velká n

$\underbrace{\frac{1}{2} \frac{\pi}{4n}}_{\text{diverguje}} \le \operatorname{tg} \frac{\pi}{4n} \le \underbrace{2 \frac{\pi}{4n}}_{\text{diverguje}}$

$\Rightarrow \sum_{n=1}^{\infty} \operatorname{tg} \frac{\pi}{4n}$ diverguje

(7) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1}-\sqrt{n-1}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1}-\sqrt{n-1}\right)\cdot\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$

$= \sum_{n=1}^{\infty} \frac{1}{n}\frac{2}{\sqrt{n+1}+\sqrt{n-1}}$

$\boxed{\lim_{n\to\infty} a_n = 0 \text{ nutná podmínka}}$

• lim. srov. kritérium s $\sum \frac{1}{n^{3/2}}$ b_k

$\lim_{n\to\infty} \frac{a_k}{b_k} = 2 \rightarrow$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ integrální kritérium

$\int_1^{\infty} \frac{1}{x^{3/2}}\,dx$, $\frac{1}{x^{3/2}}$ kladná, spojitá, nerostoucí

$\int_1^{\infty} \frac{1}{x^{3/2}} = -2\left[\frac{1}{x^{1/2}}\right]_1^{\infty} = +2 \in R \Rightarrow \sum \frac{1}{n^{2/3}}$ konverguje $\Rightarrow$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1}-\sqrt{n-1}\right)$ konverguje

(8) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ $\quad \lim_{n\to\infty} a_n = 0$

$(\ln n)^{\ln n} = e^{\ln n \ln\ln n} > e^{c\ln n}$ pro dostatečně velká n

$\ln\ln n > c$

$\ln n > e^c$

$n > e^{e^c}$

? jaké c zvolit ?

$e^{\ln n \ln\ln n} > n^c$

$\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^c} = \frac{1}{e^{c\ln n}}$

pro $c > 1$ víme z integrálního kritéria, že $\sum \frac{1}{n^c}$ konverguje, zvolme tedy např. $c = 2$

$\frac{1}{(\ln n)^{\ln n}} < \frac{1}{n^2} \quad \forall n > n_0 = e^{e^2} \approx 1619$

$\sum \frac{1}{n^2}$ konverguje a tedy $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ konverguje

(9) $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln\ln n}}$ $\quad \lim_{n\to\infty} a_n = 0$

$(\ln n)^{\ln\ln n} = e^{(\ln\ln n)^2}$

$\ln x$ roste pomaleji než x^α $\alpha > 0$

a tedy $\exists x_0 : \forall x > x_0$ $\ln x \le x^\alpha$ najdi $\alpha = \frac{1}{2}$

$$\ln x \le x^{1/2}$$

$$e^{(\ln\ln n)^2} \le e^{(\sqrt{\ln n})^2} = e^{\ln n} = n$$

a tedy $\frac{1}{e^{(\ln\ln n)^2}} \ge \frac{1}{n}$

$\sum \frac{1}{n}$ diverguje $\quad \frac{1}{(\ln n)^{\ln\ln n}} \ge \frac{1}{n}$ $\forall n > n_0$ a tedy

$\sum \frac{1}{(\ln n)^{\ln\ln n}}$ diverguje.

n^(n+1/n)

(10) $\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$

$$\lim_{n\to\infty} \frac{n^n \overbrace{\sqrt[n]{n}}^{\to 1}}{n^n (1+\frac{1}{n^2})^n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\underbrace{(1+\frac{1}{n^2})^{n^2}}_{e}}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{e}} = 1$$

→ řada diverguje, není splněna nutná podmínka konvergence

(11) Stirling $\quad n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \le n! \le e\, n^{n+\frac{1}{2}} e^{-n}$$

$\sum \frac{\ln n!}{n^\alpha}$ $\quad \alpha \in \mathbb{R}$

- $\alpha < 0$ nesplněna nutná podmínka konvergence
- $\alpha > 0$

$\ln n! = \overset{0}{\ln 1} + \ln 2 + \dots + \ln n$

$(n-1)\ln 2 \le \ln n! \le n \ln n$

$\frac{\ln n!}{n^\alpha} \le \frac{n \ln n}{n^\alpha} = \frac{\ln n}{n^{\alpha-1}}$ $\quad \alpha > 2$ konverguje (viz dále!!)

$\frac{\ln n!}{n^\alpha} \ge \frac{(n-1)\ln 2}{n^\alpha}$ diverguje pro $\alpha \le 2$ $\quad$ viz dále $\boxed{\int \frac{1}{x^\alpha (\ln x)^\beta}}$

(12) $\sum_{n=1}^{\infty} \left(n^{n^{\alpha}} - 1\right)$

$\alpha > 0$ $n^{n^\alpha} > 1$ pro $n \geq 2$ $n^{n^\alpha} - 1 \longrightarrow \neq 0$

není splněna nutná podmínka konvergence

$\alpha < 0$ $n^{n^\alpha} - 1 = e^{n^\alpha \ln n} - 1$

porovnejme s $n^\alpha \ln n$ (srovnávací limita

$$\lim_{n\to\infty} \frac{e^{\overset{\to 0}{n^\alpha} \ln n} - 1}{n^\alpha \ln n} = 1$$

$n^\alpha \to 0$ pro $\alpha < 0$ a $n \to \infty$

• ? konverguje $\sum_{n=2}^{\infty} n^\alpha \ln n$

víme $\lim_{n\to\infty} \frac{\ln n}{n^\delta} = 0$ pro $\delta > 0$

a tedy $\exists n_0$ takové, že $\ln n < n^\delta$ $\forall n \geq n_0$

• pro $-1 \leq \alpha$ diverguje dle integrálního kritéria

$\frac{\ln n}{n^{-\alpha}} \geq \frac{1}{n^{-\alpha}}$ $\frac{1}{n^{-\alpha}}$ diverguje $\forall$ $(-\alpha) \leq 1$

$\alpha \geq -1$

• pro $\alpha < -1$ $\ln n < C n^{\varepsilon}$

$\frac{\ln n}{n^{-\alpha-\varepsilon}} \frac{1}{n^{\varepsilon}} \leq \frac{C^{\in \mathbb{R}}}{n^{-\alpha-\varepsilon}}$

ε může $\frac{-\alpha-1}{2}$

a $n^{-\alpha-\varepsilon} = n^{-\frac{\alpha}{2}-\frac{1}{2}}$

a $-\frac{\alpha}{2} - \frac{1}{2} < -1$ a tedy dle integrálního kritéria

konverguje pro $\alpha < -1$

(13) $\sum\limits_{n=1}^{\infty} \left(n^{\frac{1}{n^2+1}} - 1 \right)$ $\to 0$

$$\lim_{n\to\infty} \frac{e^{\frac{1}{n^2+1}\ln n} - 1}{\frac{\ln n}{n^2+1}} = 1 \qquad \frac{\ln n}{n^2+1} \to 0$$

$\sum\limits_{n=1}^{\infty} \frac{\ln n}{n^2+1}$: $\frac{\ln n}{n^2+1} < \frac{\ln n}{n^2}$

$\sum \frac{\ln n}{n^2}$ konverguje pomocí int. kritéria $\to$

$\to \sum \frac{\ln n}{n^2+1}$ konverguje $\to \sum \left(n^{\frac{1}{n^2+1}} - 1 \right)$ konverguje

(14) $\sum\limits_{n=1}^{\infty} \frac{2\cdot 5\cdot 8 \cdot \ldots \cdot (3n-1)}{1\cdot 5\cdot 9 \cdot \ldots \cdot (4n-3)}$

$$\frac{a_{n+1}}{a_n} = \frac{2\cdot 5\cdot 8 \cdot \ldots \cdot (3n-1)(3n+2)}{1\cdot 5\cdot 9 \cdot \ldots \cdot (4n-3)(4n+1)} \cdot \frac{1\cdot 5\cdot 9 \cdot \ldots \cdot (4n-3)}{2\cdot 5\cdot 8 \cdot \ldots \cdot (3n-1)}$$

$\longrightarrow \frac{3}{4} < 1 \Rightarrow$ konverguje

(15) $\sum\limits_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$ Stirling nebo podílové

podílové

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{2^{\underbrace{(n+1)^2}_{n^2+2n+1}}} \cdot \frac{2^{n^2}}{(n!)^2} = \frac{(n+1)^2}{2^{2n+1}} \to 0 < 1 \Rightarrow \text{konverguje}$$

(16) $\sum\limits_{n=1}^{\infty} \frac{n^2}{\left(\frac{\pi}{3} + \frac{1}{n}\right)^n}$

$$\frac{\sqrt[n]{n^2}}{\frac{\pi}{3} + \frac{1}{n}} = \frac{n^{\frac{2}{n}}}{\frac{\pi}{3} + \frac{1}{n}} \to \frac{3}{\pi} < 1 \Rightarrow \text{konverguje}$$

(17) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(2n^2+n+1)^{\frac{n}{2}}}$

$\sqrt[n]{a_n} = \dfrac{n}{(2n^2+n+1)^{1/2}} = \dfrac{1}{\left(2+\frac{1}{n}+\frac{1}{n^2}\right)^{1/2}} \longrightarrow \dfrac{1}{\sqrt{2}} < 1 \Rightarrow$ konverguje

(18) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$, $\dfrac{1}{x}\dfrac{1}{(\ln x)^p}$ kladná, od nějakého x nerostoucí

$\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx = \left| \begin{array}{l} t = \ln x \\ \frac{dt}{dx} = \frac{1}{x} \end{array} \right| = \int_{\ln 2}^{\infty} \frac{1}{t^p}\,dt$

$p > 1:$ $\displaystyle\int \frac{1}{t^p}\,dt = \left[\frac{t^{-p+1}}{-p+1}\right]_{\ln 2}^{\infty}$ konverguje

$p = 1$ $\displaystyle\int_{\ln 2}^{\infty} \frac{1}{t}\,dt = \left[\ln t\right]_{\ln 2}^{\infty}$ diverguje

$p < 1$ $\displaystyle\int_{\ln 2}^{\infty} \frac{1}{t^p}\,dt = \left[\frac{t^{-p+1}}{-p+1}\right]_{\ln}^{\infty}$ diverguje

$\Rightarrow$ řada konverguje pro $p > 1$ a diverguje pro $p \leq 1$

(19) $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p(\ln\ln n)^q}$ $\quad p,q\in\mathbb{R}$

$\lim_{n\to\infty} a_n = 0$ $\forall p,q \in \mathbb{R}$ nebot'

$\alpha,\beta,\gamma>0$ $\quad x^\alpha \gg \ln^\beta x \gg \ln\ln^\gamma x$ pro $x\to\infty$

$\lim_{x\to\infty} \frac{\ln^\beta x}{x^\alpha}=0$ $\quad \exists x_0 \forall x\ge x_0$ $(\ln x)^\beta < c_\varepsilon x^\varepsilon$

$\lim_{x\to\infty}\frac{\ln\ln^\gamma x}{\ln^\beta x}=0$ $\quad \exists x_0\forall x\ge x_0$ $(\ln\ln x)^q < c_\varepsilon (\ln x)^\varepsilon$

- $p,q\le 0$

$\frac{1}{n(\ln n)^p(\ln\ln n)^q} \ge \frac1n$ $\forall n\ge n_0$ $\quad \sum\frac1n$ diverguje $\Rightarrow \sum$ diverguje

- $p>0$
  - $0<p<1$
    - $q\le 0$ jednoduchý

$\frac{1}{n(\ln n)^p(\ln\ln n)^q} \ge \frac{1}{n\ln n^p}$

$\sum \frac{1}{n(\ln n)^p}$ $\dots\to$ $\int_3^\infty \frac{1}{x(\ln x)^p} = |t=\ln x| = \int_{\ln 3}^\infty \frac{1}{t^p}\,dt \to$

$\Rightarrow$ diverguje pro $0<p\le 1$

- $q>0$ (musí pomoci ke konvergenci)

ale $\exists x_0 \forall x\ge x_0$ $(\ln\ln x)^q \le c_\varepsilon (\ln x)^\varepsilon$, $c_\varepsilon\in\mathbb{R}^+$

a $\frac{1}{n(\ln n)^p(\ln\ln n)^q} \ge \frac{1}{c_\varepsilon}\frac{1}{n\ln n^p(\ln n)^{-\varepsilon}} =$

$= \frac{1}{c_\varepsilon}\frac{1}{n\ln n^{p-\varepsilon}}$

zvolme např.

$\varepsilon = \frac{1-p}{2}$

$\varepsilon>0$

a $p-\varepsilon = p-\frac12+\frac p2 = \frac32 p-\frac12$

$p\in(0,1)$

$p-\varepsilon = \frac32 p-\frac12 \in \left(-\frac12, 1\right)$

a už $\sum \frac{1}{c_\varepsilon}\frac{1}{n\ln n^{p-\varepsilon}}$ diverguje dle integrálního kritéria

• $p = 1$

$$\sum \frac{1}{n \ln n (\ln\ln n)^q}$$

$$\int_3^\infty \frac{1}{x \ln x (\ln\ln x)^q} = \left| t = \ln x \right| = \int_{\ln 3}^\infty \frac{1}{t (\ln t)^q} = \left| u = \ln t \right| =$$

$$= \int_{\underbrace{\ln\ln 3}_{>0}}^\infty \frac{1}{u^q} du$$

→ diverguje ako $q \leq 1$

→ konverguje ako $q > 1$

• $p > 1$

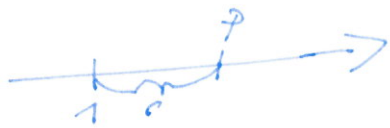
- $q > 0$ jednoduché, menší jen v konvergentní nomeri

$$\frac{1}{n (\ln n)^p (\ln\ln n)^q} \leq \frac{1}{n (\ln n)^p}$$

$$\int_3^\infty \frac{1}{x (\ln x)^p} dx = \int_{\ln 3}^\infty \frac{1}{s^p} \longrightarrow \text{konverguje } (p > 1)$$

• $q < 0$ $\quad (\ln\ln n)^{-q} < C_\varepsilon (\ln n)^\varepsilon$ ako $\forall x \geq x_0$

$$\frac{1}{n (\ln n)^p (\ln\ln n)^q} < C_\varepsilon \frac{1}{n (\ln n)^{p-\varepsilon}}$$



npr. $\varepsilon = \frac{p-1}{2}$

$p - \varepsilon = \frac{p}{2} + \frac{1}{2}$

$p \in (1, \infty)$, $\frac{p}{2} + \frac{1}{2} \in (1, \infty)$

a red $\sum \frac{1}{n (\ln n)^{p-\varepsilon}}$ konverguje

dle integrálního kritéria

celkem

$p, q \geq 0$ D

$p < 1$, q jakékoliv D

$p = 1$ $q > 1$ K

$p = 1$ $q \leq 1$ D

$p > 1$ q jakékoliv K

20 $\displaystyle\sum_{n=1}^{\infty} e^{-\sqrt[3]{n}}$

$0 \le \lim_{n \to \infty} \frac{n^{\alpha}}{e^{\sqrt[3]{n}}} \le 0 \qquad \alpha \in \mathbb{R}$

zvolme třeba $\alpha = 2$ (nebo jakékoliv $\alpha > 1$)
$\alpha = 2$ je nahodilná

$\frac{1}{e^{\sqrt[3]{n}}} \le \frac{1}{n^2}$ pro $\forall n \ge n_0$ $\quad 8 \le \frac{n_0}{\ln^3 n_0}$

$n^2 \le e^{\sqrt[3]{n}}$ platí $\frac{1}{e^{\sqrt[3]{n}}} \le \frac{1}{n^2}$

$e^{2 \ln n} \le e^{\sqrt[3]{n}}$

$8 \le \frac{n}{\ln^3 n}$

$\sum \frac{1}{n^2}$ konverguje $\Rightarrow$

$\sum \frac{1}{e^{\sqrt[3]{n}}}$ konverguje

(21) $\sum_{n=1}^{\infty} \frac{p(p+1)\dots(p+n-1)}{n!} \frac{1}{n^q} \quad p, q \in \mathbb{R}$

Raabe + podílové, Gaussovo pro $p=q$

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)^q}{n^q} \frac{(n+1)!}{n!} \frac{p(p+1)\dots(p+n-1)}{p(p+1)\dots(p+n-1)(p+n)} =$$

$$= \frac{(n+1)^q(n+1)}{n^q(p+n)} = \frac{(n+1)^{q+1}}{n^q(p+n)} = 1 + \frac{(n+1)^{q+1} - n^q(p+n)}{n^q(p+n)} =$$

$$= 1 + \frac{n^{q+1}\left(1+\frac{1}{n}\right)^{q+1} - pn^q - n^{q+1}}{n^{q+1}(p+n)} =$$

$$= 1 + \frac{n^{q+1}\left(1 + \frac{q+1}{n} + \frac{(q+1)q}{2}\frac{1}{n^2} + \dots\right) - pn^q - n^{q+1}}{n^q(p+n)}$$

alternativní řada $q\frac{1}{n^2} + \alpha_3\frac{1}{n^3}\dots$

$$= 1 + \frac{q+1+p}{n+p} + \frac{1}{n(n+p)} \underbrace{\frac{(q+1)q}{2}\left(1+\frac{1}{n}\right)^{q-1}}_{O\left(\frac{1}{n^2}\right)}$$

$$\frac{q+1+p}{n+p} = \frac{q+1-p}{n} - \frac{q+1-p}{n} + \frac{q+1-p}{n+p}$$

$$= \frac{q+1-p}{n} + (q+1-p)\left(\frac{1}{n+p} - \frac{1}{n}\right) =$$

$$\frac{n-n-p}{n(n+p)}$$

$$= \frac{q+1-p}{n} + \frac{(q+1-p)(-p)}{n(n+p)}$$

Gauss

$$\Rightarrow \frac{a_n}{a_{n+1}} = 1 + \frac{\overbrace{q+1-p}^{q}}{n} + \overbrace{\frac{(q+1-p)(-p)}{n(n+p)} + \frac{1}{n(n+p)}\frac{(q+1)q}{2}\left(1+\frac{1}{n}\right)^{q-1}}^{\text{je omez. } |c_k| < C}$$

$n^{1+\varepsilon} \quad \varepsilon > 0$

$$n\left(\frac{a_n}{a_{n-1}} - 1\right) = q+1-p + \frac{(q+1-p)(-p)}{n+p} + \dots$$

$q+1-p > 1$ konverguje $\qquad q = p$ diverguje (dle Gausse)

$q+1-p < 1$ diverguje

(22) $\displaystyle\sum_{n=1}^{\infty} \Bigg(\underbrace{\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}}_{<1}\Bigg)^p \quad p\in\mathbb{R}$

pokud $p<0$ nekonverguje (není splněna nutná podmínka)

$$\frac{a_n}{a_{n+1}} = \left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{2\cdot 4\cdots 2n(2n+2)}{1\cdot 3\cdots(2n-1)(2n+1)}\right)^p = \left(\frac{2n+2}{2n+1}\right)^p =$$

$$= \left(1+\frac{1}{2n+1}\right)^p = 1+\frac{p}{2n+1}+\underbrace{O\left(\frac{1}{n^2}\right)}_{\text{Taylor}}$$

$$= 1+\frac{p}{n\left(2+\frac{1}{n}\right)}+O\left(\frac{1}{n^2}\right) = \underbrace{1}_{h}+\underbrace{\frac{p}{2}\cdot\frac{1}{n}}_{\alpha}+O\left(\frac{1}{n^2}\right)$$

$p>2$ konverguje

$p<2$ diverguje

$p=2$ diverguje (dle Gausse)