Matematická analýza pro fyziky II

LS 2015/16, MFF UK

4. Cvičení

1. Řešte následující rovnice v okolí počátku rozvojem do řady

   a) $xy + e^x = y$,

   b) $\ln(1+x) - xy = y$,

   c) $(1+x^2)y'' - 2xy' + 2y = 0, y(0) = 0, y'(0) = 1$

   d) $y'' - xy' - 2y = 0, y(0) = 0, y'(0) = 1$

   e) $xy'' + 2y' + xy = 0, y(0) = 1$

   f) $xy'' - xy' - y = 0, y'(0) = 1$

   g) $x^2y'' - x^2y' + (x-2)y = 0, y''(0) = 2$

Řešte následující rovnice v okolí počátku rozvojem do řady

a) $xy + e^x = y$,

Obviously, $y(x) = e^x/(1-x)$, but let us find the solution by series.

Put $y(x) = \sum_{n=0}^{\infty} a_n x^n$, get

$$\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} a_n x^n,$$

$$1 - a_0 + \sum_{n=1}^{\infty} x^n \Big(a_{n-1} + \frac{1}{n!} - a_n\Big) = 0,$$

$$a_0 = 1, a_0 + 1 - a_1 = 0, a_1 + \frac{1}{2!} - a_2 = 0, \ldots, a_{n-1} + \frac{1}{n!} - a_n = 0$$

$$a_0 = 1, a_1 = 2, a_2 = 1 + 1 + \frac{1}{2!}, a_3 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!}, \ldots, a_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

and

$$y(x) = 1 + 2x + \sum_{n=2}^{\infty} \Big(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\Big) x^n$$

This is indeed the obvious solution:

$$\frac{e^x}{1-x} = \Big(\sum_{n=0}^{\infty} \frac{x^n}{n!}\Big) \cdot \Big(\sum_{n=0}^{\infty} x^n\Big) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \frac{1}{k!} \cdot 1\Big) = y(x).$$

b) $\ln(1+x) - xy = y$,

Obviously, $y(x) = \ln(1+x)/(1+x)$ is a solution. Let us recover it be series expansion: $y(x) = \sum_{n=1}^{\infty} a_n x^n$.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} a_n x^n,$$

$$-a_0 + \sum_{n=1}^{\infty} x^n \Big(\frac{(-1)^{n+1}}{n} - a_{n-1} - a_n\Big) = 0$$

$$a_0 = 0, a_1 = 1, a_2 = -\frac{1}{2} - 1, a_3 = \frac{1}{3} + \frac{1}{2} + 1, \ldots, a_n = (-1)^{n+1}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{n}\Big).$$

$$y(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\Big(1+\frac{1}{2}+\cdots+\frac{1}{n}\Big)x^n.$$

This is indeed the trivial solution:

$$\begin{aligned}
\ln(1+x)\cdot\frac{1}{1+x} &= \Big(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n\Big)\cdot\Big(\sum_{n=0}^{\infty}(-1)^n x^n\Big)\\
&= \sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}\frac{(-1)^{k+1}x^k}{k}\cdot(-1)^{n-k}x^{n-k}\Big)\\
&= \sum_{n=1}^{\infty}(-1)^{n+1}x^n\cdot\sum_{k=1}^{n}\frac{1}{k}.
\end{aligned}$$

c) $(1+x^2)y''-2xy'+2y=0, y(0)=0, y'(0)=1$

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty}a_n x^n,\\
y'(x) &= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n,\\
y''(x) &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.
\end{aligned}$$

$$\begin{aligned}
0 &= (1+x^2)y''-2xy'+2y\\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2}-\sum_{n=0}^{\infty}2(n+1)a_{n+1}x^{n+1}+\sum_{n=0}^{\infty}2a_nx^n\\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=1}^{\infty}2na_nx^n+\sum_{n=0}^{\infty}2a_nx^n\\
&= 2a_2+2a_0+6a_3x-2a_1x+2a_1x+\sum_{n=2}^{\infty}x^n\Big((n+2)(n+1)a_{n+2}+n(n-1)a_n-2na_n+2a_n\Big)
\end{aligned}$$

The initial conditions give $a_0=0, a_1=1$. Then $a_2+a_0=0$, i.e. $a_2=0$, $a_3=0$, $a_4=a_5=\cdots=0$ and $y(x)=x$.

d) $y''-xy'-2y=0, y(0)=0, y'(0)=1$

$$\begin{aligned}
0 = y''-xy'-2y &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1}-2\sum_{n=0}^{\infty}a_nx^n\\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=1}^{\infty}na_nx^n-2\sum_{n=0}^{\infty}a_nx^n\\
&= a_2-2a_0+\sum_{n=1}^{\infty}x^n\Big((n+2)(n+1)a_{n+2}-na_n-2a_n\Big)
\end{aligned}$$

By initial conditions, $a_0=0, a_1=1$, hence $a_2=0$ and $a_{n+2}=a_n/(n+1)$, $n=1,2,3,\ldots$. We get

$a_4 = a_6 = a_8 = \cdots = 0$ and $a_3 = \frac{1}{2}$, $a_5 = \frac{1}{2\cdot 4}$, $a_7 = \frac{1}{2\cdot 4\cdot 6}, \ldots$

$$\begin{aligned} y(x) &= x + \frac{1}{2}x^3 + \frac{1}{2\cdot 4}x^5 + \frac{1}{2\cdot 4\cdot 6}x^7 + \ldots \\ &= x\Big(1 + \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 + \frac{1}{2\cdot 4\cdot 6}x^6 + \ldots\Big) \\ &= x\Big(1 + \frac{x^2}{2} + \frac{(x^2/2)^2}{1\cdot 2} + \frac{(x^2/2)^3}{1\cdot 2\cdot 3} + \ldots\Big) = xe^{x^2/2}. \end{aligned}$$

e) $xy'' + 2y' + xy = 0, y(0) = 1$

$$\begin{aligned} 0 &= xy'' + 2y' + xy \\ 0 &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^{n+1} \\ 0 &= \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n + 2\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= 2a_1 + \sum_{n=1}^{\infty} x^n\Big((n+1)na_{n+1} + 2(n+1)a_{n+1} + a_{n-1}\Big) \\ &= 2a_1 + \sum_{n=1}^{\infty} x^n\Big((n+1)(n+2)a_{n+1} + a_{n-1}\Big) \end{aligned}$$

Hence, $a_0 = y(0) = 1$, $a_1 = 0$, $a_2 = -\frac{a_0}{2\cdot 3} = \frac{-1}{2\cdot 3}$, $a_3 = 0, a_4 = -\frac{a_2}{4\cdot 5} = \frac{1}{5!}, \ldots$

$$y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = \frac{\sin x}{x}.$$

f) $xy'' - xy' - y = 0, y'(0) = 1$

$$\begin{aligned} 0 = xy'' - xy' - y &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+1} - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= -a_0 + \sum_{n=1}^{\infty} x^n\Big((n+1)na_{n+1} - na_n - a_n\Big) = -a_0 + \sum_{n=1}^{\infty} x^n\Big((n+1)na_{n+1} - (n+1)a_n\Big) \end{aligned}$$

Hence, $a_0 = 0, a_1 = y'(0) = 1, a_2 = a_1 = 1, a_3 = a_2/2 = 1/2, a_4 = a_3/3 = 1/(3!), \ldots, a_n = 1/(n-1)!$

$$y(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = xe^x.$$

g) $x^2y'' - x^2y' + (x-2)y = 0, y''(0) = 2$

$$
\begin{aligned}
0 = x^2y'' - x^2y' + (x-2)y &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^{n+2} - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+2} + \sum_{n=0}^{\infty}a_nx^{n+1} - 2\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=2}^{\infty}n(n-1)a_nx^n - \sum_{n=2}^{\infty}(n-1)a_{n-1}x^n + \sum_{n=1}^{\infty}a_{n-1}x^n - 2\sum_{n=0}^{\infty}a_nx^n \\
&= -2a_0 + x(a_0 - 2a_1) + \sum_{n=2}^{\infty}x^n\Big(n(n-1)a_n - (n-1)a_{n-1} + a_{n-1} - 2a_n\Big) \\
&= -2a_0 + x(a_0 - 2a_1) + \sum_{n=2}^{\infty}x^n\Big((n-2)(n+1)a_n - (n-2)a_{n-1}\Big)
\end{aligned}
$$

We get $a_2 = y''(0)/2 = 1$, $a_0 = a_1 = 0$, $a_3 = a_2/4 = 1/4$, $a_4 = a_3/5 = 1/(4\cdot 5)$, $a_5 = a_4/6 = 1/(4\cdot 5\cdot 6), \ldots$

$$
\begin{aligned}
y(x) &= x^2 + \frac{x^3}{4} + \frac{x^4}{4\cdot 5} + \frac{x^5}{4\cdot 5\cdot 6} + \ldots \\
&= \frac{1}{x}\Big(x^3 + \frac{x^4}{4} + \frac{x^5}{4\cdot 5} + \frac{x^6}{4\cdot 5\cdot 6} + \ldots\Big) \\
&= \frac{6}{x}\Big(\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots\Big) \\
&= \frac{6}{x}\Big(e^x - 1 - x - \frac{x^2}{2}\Big) \\
&= 6\Big(\frac{e^x - 1}{x} - 1 - \frac{x}{2}\Big)
\end{aligned}
$$

2. Řešte diferenciální rovnice - separace proměnných

a) $x(x+1)y(y+1) - y' = 0; \quad y(0) = -1$
b) $y' = \frac{2xy^2}{1-x^2}, \quad y(0) = 1$
c) $y' = \frac{y}{x}, \quad x > 0$
d) $y' = \frac{y^2}{x^2}, \quad x > 0$
e) $x\sqrt{1-y^2} + y\sqrt{1-x^2}y' = 0$
f) $2\sqrt{y} = y'$
g) $(1+x^2)(1+y^2)y' + 2xy(1-y^2) = 0, \quad (0,-2)$
h) $y'\tan(x) - y = 1, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \big(\frac{\pi}{6}, 0\big)$
i) $y' = \frac{x}{y}$
j) $y' = \frac{y}{x}$
k) $y' = \tan x \tan y$ s $x, y \neq (2k+1)\frac{\pi}{2}$ a $k \in \mathbb{N}$
l) $y' + 2xy = 0$

Lösen Sie die Differentialgleichungen - Trennung der Variablen

a) $x(x+1)y(y+1) - y' = 0; \quad y(0) = -1$

$$y_1 = 0, \quad y_2 = -1$$

$$\begin{aligned}
\int \frac{\mathrm{d}y}{y(y+1)} &= \int x(x+1)\mathrm{d}x \\
\int \left(\frac{1}{y} - \frac{1}{y+1}\right)\mathrm{d}y &= \frac{x^3}{3} + \frac{x^2}{2} + c' \\
\ln\frac{|y|}{|y+1|} &= \frac{x^3}{3} + \frac{x^2}{2} + c' \\
\frac{y}{y+1} &= c\exp\{\frac{x^3}{3} + \frac{x^2}{2}\} \\
\frac{1}{y+1} &= 1 - c\exp\{\frac{x^3}{3} + \frac{x^2}{2}\} \\
y &= \frac{1}{1 - c\exp\{\frac{x^3}{3} + \frac{x^2}{2}\}} - 1
\end{aligned}$$

b) $y' = \frac{2xy^2}{1-x^2}, \quad y(0) = 1$

$$x \neq \pm 1, \quad y \neq 0$$

$y = 0$ ist Lösung der Differentialgleichung, aber AB nicht in 1.
FĂĽr $-1 < x < 1$: (AB: $x = 0 \wedge y = 1 \curvearrowright c = -1$)

$$\begin{aligned}
\int \frac{\mathrm{d}y}{y^2} &= 2\int \frac{x}{1-x^2}\mathrm{d}x \\
\frac{-1}{y} &= 1 + \ln(1-x^2) \\
y &= \frac{1}{1+\ln(1-x^2)}
\end{aligned}$$

c) $y' = \frac{y}{x}, \quad x > 0$
$y = 0$ ist Lösung. $y \neq 0$:

$$\begin{aligned}
\frac{\mathrm{d}y}{y} &= \frac{\mathrm{d}x}{x} \\
\ln|y| &= \ln x + c \\
y &= cx
\end{aligned}$$

d) $y' = \frac{y^2}{x^2}, \quad x > 0$
$y = 0$ ist Lösung. $y \neq 0$:

$$\begin{aligned}
\frac{\mathrm{d}x}{x^2} &= \frac{\mathrm{d}y}{y^2} \\
-\frac{1}{x} &= -\frac{1}{y} + c \\
\frac{1}{x} + c &= \frac{1}{y} \\
y &= \frac{1}{\frac{1}{x} + c} = \frac{x}{1+cx}
\end{aligned}$$

e) $x\sqrt{1-y^2} + y\sqrt{1-x^2}y' = 0$
Lösung $y = \pm 1$:

$$-\frac{x}{\sqrt{1-x^2}}\mathrm{d}x = \frac{y}{\sqrt{1-y^2}}$$

Lösung $y \neq 1$:

$$-\sqrt{1-x^2} = \sqrt{1-y^2} + c$$

f) $2\sqrt{y} = y'$
Lösung ist $y = 0$.

$$\frac{\mathrm{d}y}{2\sqrt{y}} = \mathrm{d}x$$
$$\sqrt{y} = x + c$$
$$y = (x+c)^2$$

g) $(1+x^2)(1+y^2)y' + 2xy(1-y^2) = 0, \quad (0,-2)$
Lösungen $y = 0, \pm 1$:

$$\frac{1+y^2}{y(y^2-1)} = \frac{2x}{1+x^2}\mathrm{d}x$$

sonst:

$$\frac{\frac{(1+y^2)}{y^2}}{\frac{(y^2-1)}{y}}\mathrm{d}y = \frac{2x}{1+x^2}\mathrm{d}x$$
$$\frac{y^2-1}{y} = c(x^2+1)$$

speziell: $\frac{4-1}{-2} = c \; c = -\frac{3}{2}$

$$y - \frac{1}{y} = -\frac{3}{2}(x^2+1)$$

h) $y' \tan(x) - y = 1, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad \left(\frac{\pi}{6}, 0\right)$
Lösung $y = -1$:

$$\frac{\mathrm{d}y}{1+y} = \frac{\mathrm{d}x}{\tan x}$$

sonst:

$$\ln|1+y| = \ln|\sin x|$$
$$1 + y = c \sin x$$

speziell: $1 = c \sin\frac{\pi}{6} = \frac{c}{2} \; c = 2$

$$y = -1 + 2\sin x$$

i) $y' = \frac{x}{y}$

$$yy' = x$$
$$\int y\mathrm{d}y = \int x\mathrm{d}x$$
$$\frac{y^2}{2} + c = \frac{x^2}{2} + c$$
$$x^2 - y^2 = c$$

j) $y' = \frac{y}{x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}$$
$$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x}$$
$$\ln|y| = \ln|x| + c$$
$$y = cx$$

k) $y' = \tan x \tan y$ mit $x, y \neq (2k+1)\frac{\pi}{2}$ und $k \in \mathbb{N}$
Ausnahmelösung: $y_k = k\pi$.

$$\begin{aligned}
\int \frac{\cos y}{\sin y} \mathrm{d}y &= \int \frac{\sin x}{\cos x} \mathrm{d}x \\
\ln|\sin y| &= -\ln|\cos x| + c' \\
\sin y &= \frac{c}{\cos x} \\
y &= \arcsin\left(\frac{c}{\cos x}\right)
\end{aligned}$$

l) $y' + 2xy = 0$
Ausnahmelösung: $y_1 = 0$.

$$\begin{aligned}
\int \frac{\mathrm{d}y}{y} &= -\int 2x \mathrm{d}x \\
\ln|y| &= -x^2 + c' \\
y &= ce^{-x^2}
\end{aligned}$$

3. Řešte diferenciální rovnice - substituce

a) $y' = \frac{y}{x} + \sin\frac{y}{x} \quad y(1) = \frac{\pi}{2}$
b) $y + (2\sqrt{xy} - x)y' = 0 \quad y(2) = 0$
c) $y^2 - 4xy + 4x^2y' = 0$
d) $y' = \frac{y}{x} + \sin\frac{y}{x}$
e) $xy' = y\ln\left(\frac{y}{x}\right)$
f) $xyy' = x^2 + y^2$ s počáteční podmínkou $y(1) = 1$
g) $x^2 + xy + y^2 = x^2y'$
h) $y' = \frac{2xy}{x^2+y^2}$
i) $y' = \frac{x+2y}{x}$
j) $y' = \frac{y}{x+y}$

Lösen Sie die Differentialgleichungen - Substitution

a) $y' = \frac{y}{x} + \sin\frac{y}{x} \quad y(1) = \frac{\pi}{2}$

$$u = \frac{y}{x} \quad u' = \frac{1}{x}(u + \sin(u) - u) = \frac{\sin u}{x}$$

Ausnahmelösung: $\sin u = 0 \quad u = k\pi \quad y = k\pi x$.

$$\begin{aligned}
\int \frac{\mathrm{d}u}{\sin u} &= \ln|x| + c' \\
\int \frac{\mathrm{d}u}{2\sin\frac{u}{2}\cos\frac{u}{2}} &= \ln|x| + c' \\
\int \frac{\mathrm{d}u}{2\tan\frac{u}{2}\cos^2\frac{u}{2}} &= \ln|x| + c' \\
\ln\left|\tan\frac{u}{2}\right| &= \ln|x| + c' \\
x &= \tan\frac{y}{2x}
\end{aligned}$$

b) $y+(2\sqrt{xy}-x)y'=0 \quad y(2)=0$

$$
\begin{aligned}
y' &= \frac{y}{x-2\sqrt{xy}}\\
y' &= \frac{u}{1-\sqrt{u}}\\
\text{mit}\quad u=\frac{y}{x}\quad u' &= \frac{xy'-y}{x^2}=\frac{1}{x}\frac{2u\sqrt{u}}{1-2\sqrt{u}}\\
\int\left(\frac{1}{2u\sqrt{u}}-\frac{1}{u}\right)\mathrm{d}u &= \ln|x|+c'\\
-\frac{1}{\sqrt{u}}-\ln|u| &= \ln|x|+c'\\
-\sqrt{\frac{x}{y}}-\ln\left(\frac{y}{x}\right) &= \ln|x|+c'\\
ye^{\sqrt{\frac{x}{y}}} &= c
\end{aligned}
$$

c) $y^2-4xy+4x^2y'=0$

$$
\begin{aligned}
y' &= \frac{4xy-y^2}{4x^2}=\frac{4u-u^2}{4}=u-\frac{u^2}{4}\quad u=\frac{y}{x}\\
u' &= \frac{y'x-y}{x^2}=\frac{1}{x}\left(u-\frac{u^2}{4}-u\right)=\frac{-u^2}{4x}\\
-\int\frac{\mathrm{d}u}{u^2} &= \frac{1}{4}\ln|x|+c'\\
\frac{1}{u} &= \frac{1}{4}\ln|x|+c'\\
\frac{4x}{y} &= \ln|x|+c'\\
y &= \frac{4x}{\ln|x|+c}
\end{aligned}
$$

d) $y'=\frac{y}{x}+\sin\frac{y}{x}$

Substitution: $z\frac{y}{x}$, $z'=\frac{y'x-y}{x^2}=\frac{\sin z}{x}$

$$
\begin{aligned}
\int\frac{\mathrm{d}z}{\sin z} &= \int\frac{\mathrm{d}x}{x}\\
\frac{1}{2}\int\frac{\sin^2\frac{z}{2}+\cos^2\frac{z}{2}}{\sin\frac{z}{2}\cos\frac{z}{2}}\mathrm{d}z &= \ln|x|+c\\
-\ln\left|\cos\frac{z}{2}\right|+\ln\left|\sin\frac{z}{2}\right|+c &= \ln|x|+c\\
\ln\left|\tan\frac{z}{2}\right|+c &= \ln|x|+c\\
\tan\frac{z}{2} &= cx\\
\frac{z}{2} &= \arctan(cx)+k\pi\\
y=2x[\arctan(cx)+k\pi]
\end{aligned}
$$

Sonderlösung: $y=k\pi x$.

e) $xy'=y\ln\left(\frac{y}{x}\right)$

Substitution: $z(x) = \frac{y(x)}{x}$ und $z' = \frac{1}{x}\left(y' - \frac{y}{x}\right) = \frac{1}{x}(z\ln(z) - z)$.

$$\begin{aligned}
\int \frac{\mathrm{d}z}{z(\ln(z)-1)} &= \int \frac{\mathrm{d}x}{x} \\
\ln|\ln z - 1| &= \ln|x| + c \\
\ln\frac{y}{x} - 1 &= cx \\
\ln y &= cx + 1 + \ln|x| \\
y &= xee^{cx}
\end{aligned}$$

f) $xyy' = x^2 + y^2$ mit der Anfangsbedingung $y(1) = 1$ durch eine geeignete Substitution

Substitution: $z(x) = \frac{y(x)}{x}$ und $z' = \frac{1}{x}\left(y' - \frac{y}{x}\right) = \frac{1}{zx}$.

$$\begin{aligned}
\int z\mathrm{d}z &= \frac{\mathrm{d}x}{x} \\
\frac{z^2}{2} &= \ln|x| + c \\
y^2 &= 2x^2(\ln|x| + c)
\end{aligned}$$

Mit Anfangsbedingung:

$$\begin{aligned}
1 = 2(\ln 1 + c) &= 2c \quad c = \frac{1}{2} \\
y^2 = 2x^2\left(\frac{1}{2} + \ln|x|\right)
\end{aligned}$$

g) $x^2 + xy + y^2 = x^2y'$

Substitution: $z = \frac{y}{x}$ und $z' = \frac{y'x - y}{x^2} = \frac{y' - z}{x} = \frac{1+z^2}{x}$ mit $y' = 1 + z + z^2$.

$$\begin{aligned}
\int \frac{\mathrm{d}z}{1+z^2} &= \int \frac{\mathrm{d}x}{x} \\
\arctan z &= \ln|x| + c \\
\arctan\frac{y}{x} &= \ln|x| + c
\end{aligned}$$

h) $y' = \frac{2xy}{x^2+y^2}$

Substitution: $z = \frac{y}{x}$ und $z' = \frac{1}{x}\left(\frac{2z}{1+z^2} - z\right) = \frac{z - z^3}{1+z^2}$ mit $y' = \frac{2\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}$.

$$\begin{aligned}
\int \left(\frac{1}{z} - \frac{1}{z-1} - \frac{1}{z+1}\right)\mathrm{d}z &= \int \frac{\mathrm{d}x}{x} \\
\ln\left|\frac{z}{z^2-1}\right| &= \ln|x| + c \\
\frac{z}{z^2-1} &= cx \\
cy &= y^2 - x^2
\end{aligned}$$

i) $y' = \frac{x+2y}{x}$

Substitution: $z = \frac{y}{x}$ und $z' = \frac{y'-z}{x} = \frac{1}{x}(1+z)$ mit $y' = 1 + 2\frac{y}{x}$.

$$\begin{aligned}
\int \frac{\mathrm{d}z}{z+1} &= \int \frac{\mathrm{d}x}{x} \\
\ln|z+1| &= \ln|x| + c \\
z + 1 &= cx \\
\frac{y}{x} + 1 &= xc \\
y = -x + cx^2
\end{aligned}$$

j) $y' = \frac{y}{x+y}$

Substitution: $z = \frac{y}{x}$ und $z' = \frac{y'x-y}{x^2} = \frac{z^2}{1+z}$ mit $y' = 1 - \frac{1}{1+\frac{y}{x}}$.

$$\begin{aligned} \int \frac{1+z}{z^2}\mathrm{d}z &= \int \frac{\mathrm{d}x}{x} \\ -\frac{1}{z} + \ln|z| &= -\ln|x| + c \\ -\frac{x}{y} + \ln|y| &= c \\ x &= y(\ln|y| + c) \end{aligned}$$

4. Řešte diferenciální rovnice - variace konstant

a) $x^3 + y - 2xy' = 0$

b) $xy' - 2y = e^x(x-2)$

c) $y' - y\cos x = 3\cos x$

d) $xy' + 2y = x^2$

e) $y' = y\tan(x) + 1$ s počáteční podmínkou $y\left(\frac{\pi}{4}\right) = 1 + \sqrt{2}$

f) $y' - \frac{y}{x} = -\sqrt{x}$ s počáteční podmínkou $y(1) = 3$

g) $(1+x^2)y' + xy = 1$ s počáteční podmínkou $y(0) = 1$.

h) $y' + \frac{1}{2x}y = \sqrt{x}\sin(x)$ s $y(\pi) = 2\sqrt{\pi}$

i) $(1+x^2)y' - 2xy = (1+x^2)^2$

j) $y' = \frac{\sin x}{\cos x}y + \cos x$

Lösen Sie die Differentialgleichungen - Variation der Konstanten

a) $x^3 + y - 2xy' = 0$

Homogene Lösung:

$$\begin{aligned} y' - \frac{y}{2x} &= \frac{x^2}{2} \\ \int \frac{\mathrm{d}y}{y} &= \frac{1}{2}\int \frac{\mathrm{d}x}{x} \\ y &= c\sqrt{|x|} \end{aligned}$$

Spezielle Lösung mit Variation der Konstanten.

$$\begin{aligned} c'\sqrt{|x|} &= \frac{x^2}{2} \\ c' &= \begin{cases} \frac{x^{3/2}}{2} & x > 0 \\ \frac{(-x)^{3/2}}{2} & x < 0 \end{cases} \\ c &= \begin{cases} \frac{x^{5/2}}{2} & x > 0 \\ \frac{(-x)^{5/2}}{2} & x < 0 \end{cases} \end{aligned}$$

Spezielle inhomogene Lösung:

$$y = \frac{x^3}{5}$$

Allgemeine inhomogene Lösung:

$$y = c\sqrt{|x|} + \frac{x^3}{5}$$

b) $xy' - 2y = e^x(x-2)$
Homogene Lösung:

$$\frac{\mathrm{d}y}{y} = \frac{\mathrm{d}x}{x}$$
$$\ln|y| = 2\ln|x|$$
$$y = cx^2$$

Spezielle Lösung durch Variation der Konstanten mit $y(x) = c(x)x^2$.

$$c'x^3 + 2cx^2 - 2cx^2 = e^x(x-2)$$
$$c'(x) = \left(\frac{1}{x^2} - \frac{2}{x^3}\right)e^x$$
$$c(x) = \frac{e^x}{x^2}$$

Allgemeine Lösung:

$$y(x) = cx^2 + e^x$$

c) $y' - y\cos x = 3\cos x$
Homogene Lösung:

$$y' = y\cos x$$
$$\int \frac{\mathrm{d}y}{y} = \int \cos(x)\mathrm{d}x$$
$$\ln|x| = \sin(x) + c$$
$$y = ce^{\sin x}$$

Spezielle Lösung mit Variation der Konstanten.

$$c\left(e^{\sin x}\right)' + c'e^{\sin x} = ce^{\sin x}\cos x + 3\cos x$$
$$c' = 3\cos(x)e^{-\sin x}$$
$$c = -3e^{-\sin x}$$

Spezielle Lösung:

$$y = -3$$

Allgemeine Lösung:

$$y = ce^{\sin x} - 3$$

d) $xy' + 2y = x^2$
Homogene Lösung:

$$y' = -2\frac{y}{x}$$
$$\frac{\mathrm{d}y}{2y} = -\frac{\mathrm{d}x}{x}$$
$$\ln|y| = -2\ln|x| + c$$
$$y = \frac{c}{x^2}$$

Spezielle Lösung durch Variation der Konstanten mit $y(x) = \frac{c(x)}{x^2}$.

$$\frac{c'x - 2c}{x^2} + \frac{2c}{x^2} = x^2$$
$$\frac{c'}{x} = x^2$$
$$c = \frac{x^4}{4}$$

Allgemeine Lösung:

$$y(x) = \frac{c}{x^2} + \frac{x^4}{4}$$

e) $y' = y\tan(x) + 1$ mit der Anfangsbedingung $y\left(\frac{\pi}{4}\right) = 1 + \sqrt{2}$
Homogene Lösung:

$$\begin{aligned} \int \frac{\mathrm{d}y}{y} &= \int \tan(x)\mathrm{d}x \\ \ln|y| &= -\ln|\cos x| \\ y &= \frac{c}{\cos x} \end{aligned}$$

Spezielle Lösung mit Variation der Konstanten.

$$\begin{aligned} \frac{c'(x)}{\cos x} &= 1 \\ c(x) &= \sin x \end{aligned}$$

Spezielle Lösung:

$$y = \frac{\sin x}{\cos x} = \tan x$$

Allgemeine Lösung:

$$y = \frac{c}{\cos x} + \tan x$$

Einsetzen der Anfangsbedingung:

$$1 + \sqrt{2} = \frac{c}{\sqrt{2}} + 1 = c\sqrt{2} + 1, \quad c = 1$$

Lösung des Anfangswertproblems:

$$y = \frac{1 + \sin x}{\cos x}$$

f) $y' - \frac{y}{x} = -\sqrt{x}$ mit der Anfangsbedingung $y(1) = 3$
Homogene Lösung:

$$\begin{aligned} \int \frac{\mathrm{d}y}{y} &= \int \frac{\mathrm{d}x}{x} \\ \ln|y| &= \ln|x| + c \\ y &= cx \end{aligned}$$

Spezielle Lösung mit Variation der Konstanten.

$$\begin{aligned} y(x) &= c(x)x \\ c' &= -\sqrt{x} \\ c(x) &= -2x^{\frac{1}{2}} \end{aligned}$$

Allgemeine Lösung:

$$y(x) = cx - 2\sqrt{x}^3$$

Einsetzen der Anfangsbedingung:

$$3 = c - 2, \quad c = 5$$

Lösung des Anfangswertproblems:

$$y(x) = 5x - 2\sqrt{x}^3$$

g) $(1+x^2)y' + xy = 1$ mit der Anfangsbedingung $y(0) = 1$.
Homogene Lösung:

$$\int \frac{\mathrm{d}y}{y} = \int \frac{x\mathrm{d}x}{x^2-1}$$
$$\ln|y| = \frac{1}{2}\ln(1-x^2) + c$$
$$y = c\sqrt{1-x^2}$$

Spezielle Lösung mit Variation der Konstanten.

$$y(x) = c(x)\sqrt{1-x^2}$$
$$c'(x) = \frac{1}{\sqrt{1-x^2}^3}$$
$$c(x) = \frac{x}{\sqrt{1-x^2}}$$

Allgemeine Lösung:

$$y(x) = c\sqrt{1-x^2} + x$$

Lösung mit Anfangsbedingung:

$$y(x) = \sqrt{1-x^2} + x$$

h) $y' + \frac{1}{2x}y = \sqrt{x}\sin(x)$ mit $y(\pi) = 2\sqrt{\pi}$
Homogene Lösung:

$$\int \frac{\mathrm{d}y}{y} = -\int \frac{\mathrm{d}x}{2x}$$
$$\ln|y| = -\frac{1}{2}\ln|x| + c$$
$$y = c\sqrt{x}$$

Spezielle Lösung mit Variation der Konstanten.

$$y = \frac{c(x)}{\sqrt{x}}$$
$$\frac{c'}{\sqrt{x}} = \sqrt{x}\sin x$$
$$c(x) = -x\cos x + \sin x$$

Allgemeine Lösung:

$$y(x)\frac{c + \sin x - x\cos x}{\sqrt{x}}$$

Einsetzten der Anfangsbedingung:

$$2\sqrt{\pi} = y(\pi) = \frac{c+\pi}{\sqrt{\pi}}, \quad c = \pi$$

Lösung des Anfangswertproblems:

$$y(x)\frac{\pi + \sin x - x\cos x}{\sqrt{x}}$$

i) $(1+x^2)y' - 2xy = (1+x^2)^2$

Homogene Lösung:

$$(1+x^2)y' = 2xy$$
$$\int \frac{\mathrm{d}y}{y} = \int \frac{2x\mathrm{d}x}{1+x^2}$$
$$\ln|y| = \ln(1+x^2) + c$$
$$y = c(1+x^2)$$

Inhomogene Lösung durch Variation der Konstanten:

$$(1+x^2)c'(1+x^2) = (1+x^2)^2$$
$$c = x$$

Allgemeine Lösung:

$$y = (x+c)(1+x^2)$$

j) $y' = \frac{\sin x}{\cos x}y + \cos x$
Homogene Lösung:

$$y' = \frac{\sin x}{\cos x}y$$
$$\int \frac{\mathrm{d}y}{y} = \int \frac{\sin x}{\cos x}\mathrm{d}x$$
$$\ln|y| = -\ln|\cos x| + c$$
$$y = \frac{c}{\cos x}$$

Inhomogene Lösung durch Variation der Konstanten:

$$Y = \frac{c(x)}{\cos x}$$
$$c'(x) = \cos^2 x$$
$$c(x) = \frac{1}{2}(\sin x \cos x + x)$$

Allgemeine Lösung:

$$y = \frac{c}{\cos x} + \frac{1}{2}\sin x + \frac{1}{2}\frac{x}{\cos x}$$

5. Řešte diferenciální rovnice

a) $x^2y'' + 5xy' + 4y = 0$
b) $y'' - y = (1+x)e^{2x}$
c) $y''' + 7y' + 6y = 0$
d) $y^{(5)} + 8y''' + 16y' = 0$
e) $x^2y'' + xy' + y = 0$
f) $x^2y'' + 6xy' + 6y = e^x$ s $y(x) = -1$ a $y'(1) = 2$
g) $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$
h) $x^2y'' - 3xy' + 5y = 0$ s $y(1) = 1$ a $y'(1) = 0$
i) $x^2y'' - 4xy' + 6y = 0$
j) $x^2y'' - 3xy' + 5y = 0$
k) $y'' + 4y' + 4y = e^{-2x}\ln x, \quad x > 0$
l) $y''' - y' = e^{2x}$

Lösen Sie die Differentialgleichungen - Eulersche Differentialgleichungen

a) $x^2y'' + 5xy' + 4y = 0$
Ansatz:

$$\begin{aligned} y &= x^\lambda; \\ 0 &= \lambda(\lambda-1) + 5\lambda + 4 = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 \end{aligned}$$

Lösung:

$$y = \frac{c_1}{x^2} + c_2\frac{\ln x}{x^2}$$

Aufgabe:
Lösen Sie die Differentialgleichung

$$y'' - y = (1+x)e^{2x}.$$

b) $y'' - y = (1+x)e^{2x}$
Ansatz homogene Lösung:

$$\begin{aligned} y &= e^{\lambda x}, \\ \lambda^2 - 1 &= 0, \\ \lambda &= \pm 1 \end{aligned}$$

Homogene Lösung:

$$y = c_1e^x + c_2e^{-x}$$

Ansatz inhomogene Lösung:

$$\begin{aligned} y &= (ax+b)e^{2x} \\ y' &= (4ax + 4b + a)e^{2x}) \\ y'' &= (4ax + 4b + 2a + 2a)e^{2x} \end{aligned}$$

In DGL eingesetzt:

$$\begin{gathered} 4ax + 4b + 4a - ax - b = x + 1 \\ a = \frac{1}{3}, \quad 3b + \frac{4}{3} = 1, \quad b = -\frac{1}{9} \end{gathered}$$

Allgemein Lösung:

$$y = c_1e^x + c_2e^{-x} + \left(\frac{x}{3} - \frac{1}{9}\right)e^{2x}$$

c) $y''' + 7y' + 6y = 0$
Ansatz:

$$\begin{aligned} y(x) &= e^{\lambda x} \\ \lambda^3 - 7\lambda + 6 &= 0 \\ \lambda &= 1 \\ (\lambda^3 - 7\lambda + 6)(\lambda - 1) &= \lambda^2 + \lambda - 6 \\ \lambda_{2/3} &= -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} = -\frac{1}{2} \pm \frac{5}{2} \\ \lambda_2 = 2, &\quad \lambda_3 = -3 \end{aligned}$$

Lösung:

$$y(x) = c_1e^x + c_2e^{2x} + c_3e^{-3x}$$

d) $y^{(5)} + 8y''' + 16y' = 0$
Ansatz:

$$\begin{aligned} y &= e^{\lambda x} \\ \lambda_1 &= 0 \\ \lambda^4 + 8\lambda^2 + 16 &= 0 \\ \lambda &= \pm 2i, \quad \text{doppelt} \end{aligned}$$

Lösung:

$$y = c_1 + c_2 \cos 2x + c_3 \sin 2x + c_4 x \cos 2x + c_5 x \sin 2x$$

e) $x^2y'' + xy' + y = 0$
Ansatz:

$$\begin{aligned} y &= x^{\lambda} \\ \lambda(\lambda - 1) + \lambda + 1 &= 0 \\ \lambda^2 + 1 &= 0 \\ \lambda &= \pm i \\ x^{\pm i} &= e^{\pm i \ln |x|} = \cos \ln |x| \pm i \sin \ln |x| \end{aligned}$$

Allgemeine Lösung:

$$y(x) = c_1 \cos \ln |x| + c_2 \sin \ln |x|$$

f) $x^2y'' + 6xy' + 6y = e^x$ mit $y(x) = -1$ und $y'(1) = 2$
Ansatz: $y = x^{\lambda}$

$$\begin{aligned} \lambda(\lambda - 1) + 6\lambda + 6 &= 0 \\ \lambda_1 &= -2 \\ \lambda_2 &= -3 \end{aligned}$$

Homogene Lösung:

$$y = \frac{c_1}{x^2} + \frac{c_2}{x^3}$$

Inhomogen:

$$y'' + \frac{6}{x}y' + \frac{6}{x^2}y$$

Variation der Konstanten:

$$\begin{aligned} \frac{c_1'}{x^2} + \frac{c_2'}{x^3} &= 0 \\ \Rightarrow \quad \frac{-2c_1'}{x^3} - \frac{3c_2'}{x^4} &= \frac{e^x}{x^2} \\ 2xc_1' + 3c_2' &= x^2 e^x \\ c_1' = xe^x, &\quad c_2' - x^2 e^x \\ c_1 = (x-1)e^x, &\quad c_2 = (-x^2 + 2x - 2)e^x \end{aligned}$$

Spezielle inhomogene Lösung:

$$y = \left(\frac{1}{x^2} - \frac{2}{x^3}\right) e^x$$

Einsetzen der Anfangswerte:

$$\begin{aligned} y(1) &= c_1 + c_2 - e = -1 \\ y'(1) &= 3e - 2c_1 - 3c_2 = 2 \\ c_1 = e, &\quad c_2 = -1 \end{aligned}$$

Allgemeine Lösung:

$$y = \frac{e^x - 1}{x^2} + \frac{e - 2e^x}{x^3}$$

g) $x^3y''' - 3x^2y'' + 6xy' - 6y = 0$
Ansatz: $y(x) = x^\lambda$

$$\lambda(\lambda-1)(\lambda-2) - 3\lambda(\lambda-1) + 6\lambda - 6 = 0$$
$$\lambda_1 = 1$$
$$\text{Polynomdivision:}$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$\lambda_{2/3} = \frac{5}{2} \pm \frac{1}{2}$$

Lösung:

$$y(x) = c_1x + c_2x^2 + c_3x^3$$

h) $x^2y'' - 3xy' + 5y = 0$ mit $y(1) = 1$ und $y'(1) = 0$
Ansatz: $y = x^\lambda$

$$\lambda(\lambda-1) - 3\lambda + 5 = 0$$
$$\lambda_{1/2} = -2 \pm \sqrt{4-5} = -2 \pm i$$
$$x^{2\pm i} = x^2x^{\pm i} = x^2e^{\pm i\ln x} = x^2(\cos\ln x \pm i\sin\ln x)$$

Allgemeine Lösung:

$$y = x^2(c_1\cos\ln x + c_2\sin\ln x)$$
$$y' = \frac{2}{x}y + x(-c_1\sin\ln x + c_2\cos\ln x)1 = y(1) = c_1 \qquad \Rightarrow \qquad c_1 = 1$$
$$0 = y'(1) = 2 + c_2 \qquad \Rightarrow \qquad c_2 = -2$$

Lösung:

$$y = x^2(\cos\ln x - 2\sin\ln x)$$

i) $x^2y'' - 4xy' + 6y = 0$
Ansatz: $y = x^\lambda$

$$\lambda(\lambda-1) - 4\lambda + 6 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$\lambda_{1/2} = \frac{5}{2} \pm \frac{1}{2}$$

Allgemeine Lösung:

$$y(x) = c_1x^2 + c_2x^3$$

j) $x^2y'' - 3xy' + 5y = 0$
Ansatz: $y(x) = x^\lambda$

$$\lambda^2 - 4\lambda + 5 = 0$$
$$\lambda_{1/2} = 2 \pm i$$
$$x^{2\pm i} = x^2e^{\pm i\ln x} = x^2(\cos\ln x \pm i\sin\ln x)$$

Lösung:

$$y = x^2(c_1\cos\ln x + c_2\sin\ln x)$$

k) $y'' + 4y' + 4y = e^{-2x} \ln x, \quad x > 0$

Hommogene Lösung:

$$\lambda^2 + 4\lambda + 4 = 0 = (\lambda + 2)^2$$

Allgemeine homognene Lösung:

$$y = (c_1 + c_2 x)e^{-2x}$$

Variation der Konstaten:

$$\begin{pmatrix} e^{-2x} & xe^{-2} \\ -2e^{-2x} & (1-2x)e^{-2x} \end{pmatrix} \cdot \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-2x} \ln x \end{pmatrix}$$

$$\begin{aligned} c_1' + xc_2' &= 0 \\ -2c_1' + (1-2x)c_2' &= \ln x \\ \Rightarrow \quad c_2' = \ln x \quad &\Rightarrow \quad c_2 = x(\ln x - 1) \\ \Rightarrow \quad c_1' = -c \ln x \quad &\Rightarrow \quad c_1 = \frac{-x^2}{4}(2\ln x - 1) \end{aligned}$$

Allgemeine Lösung:

$$y = \left(c_1 + c_2 x + \frac{x^2}{2} \ln x - \frac{3}{4}x^2 \right) e^{-2x}$$

l) $y''' - y' = e^{2x}$

Ansatz homogene Lösung: $y(x) = e^{\lambda x}$

$$\begin{aligned} &\lambda^3 - \lambda = 0 \\ &\lambda_1 = 0 \quad \lambda_{2/3} = \pm 1 \end{aligned}$$

Homogene Lösung:

$$y(x) = c_1 + c_2 e^x + c_3 e^{-x}$$

Ansatz inhomogene Lösung:

$$y = be^{2x}$$

eingesetzt:

$$8b - 2b = 1 \quad b = \frac{1}{6}$$

Allgemeine Lösung:

$$y(x) = c_1 + c_2 e^x + c_3 e^{-x} + \frac{1}{6} e^{2x}$$

6. Řešte Clairantovu diferenciální rovnici

$$y = xy' + y' - y'^2$$

$$y = xy' + y'^2$$

$$y = xy' + \sqrt{1 + (y')^2}$$

Lösen Sie die Clairantsche Differentialgleichung

$$y = xy' + y' - y'^2$$

$$y = xy' + y'^2$$

$$y = xy' + \sqrt{1 + (y')^2}$$

$$\begin{aligned} y &= xz + z - z^2 \\ g(z) &= z - z^2 \\ g'(z) &= 1 - 2z \end{aligned}$$

Geradenschar: $y = cx + c - c^2$

$$\begin{aligned} x + g'(z) &= 0 \\ x + 1 - 2z &= 0 \\ z &= \frac{x+1}{2} \\ y &= x\left(\frac{x+1}{2}\right) + \frac{x+1}{2} - \frac{(x+1)^2}{4} = \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4} \\ &= \frac{(x+1)^2}{4} \end{aligned}$$

Singuläre Lösung:

$$y = \frac{(x+1)^2}{4}$$

Geradenschar: $y = cx + c^2$
Singuläre Lösung:

$$\begin{aligned} g(z) &= z^2 \\ g'(z) &= 2z \\ x + g'(z) &= 0 \\ x + 2z &= 0 \\ z &= -\frac{x}{2} \\ y &= xz + z^2 = x\left(-\frac{x}{2}\right) + \left(-\frac{x}{2}\right)^2 = -\frac{x^2}{4} \end{aligned}$$

Singuläre Lösung:

$$y = -\frac{x^2}{4}$$

Clairant: $y = xy' + f(y')$

$$\begin{aligned} 0 &= y''(x + f'(y')) \\ y(x) &= cx + f(c) \\ x &= -f'(y') \\ y' &= t \quad \text{Parameter} \\ (x, y) &= (-f'(t), -tf'(t) + f(t)) \end{aligned}$$

Geradenschar: $y(x) = cx + \sqrt{1+c^2}$ und

$$\begin{aligned} x &= -\frac{y'}{\sqrt{1+y'^2}} \\ x(t) &= -\frac{t}{\sqrt{1+t^2}} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= -\frac{1}{\sqrt{1+t^2}^3} \end{aligned}$$

Gesuchte Funktion g mit $y = g(t)$:

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}t} &= t\left(-\frac{1}{\sqrt{1+t^2}^3}\right) = -\frac{t}{\sqrt{1+t^2}^3} \\
g(t) &= \frac{1}{\sqrt{1+t^2}} \\
x(t) &= -\frac{t}{\sqrt{1+t^2}} \\
y(t) &= \frac{1}{\sqrt{1+t^2}} \\
\Rightarrow \quad x^2 + y^2 &= 1
\end{aligned}$$

7. Řešte d'Alembertovu diferenciální rovnici

$$y = x(1+y') + y'^2$$

$$y = \frac{xy'^2}{2(y'+2)}$$

$$y = x + (y')^3 - 3y'$$

Lösen Sie die d'Alembertsche Differentialgleichung

$$y = x(1+y') + y'^2$$

$$y = \frac{xy'^2}{2(y'+2)}$$

$$y = x + (y')^3 - 3y'$$

$$\begin{aligned}
y &= f(y')x + g(y') \\
f(z) &= 1 + z \\
g(z) &= z^2
\end{aligned}$$

keine singuläre Lösung.

$$\begin{aligned}
\frac{f'(z)}{z - f(z)} &= \frac{1}{-1} = -1 \\
\frac{g'(z)}{z - f(z)} &= \frac{2z}{-1} = -2z \\
\frac{\mathrm{d}x}{\mathrm{d}z} &= -x - 2z \\
x &= ce^{-z} + 2 - 2z \\
& y(ce^{z} + 2 - 2z)(1+z) + z^2 = c(1+z)e^{-z} + 2 - z^2
\end{aligned}$$

Parameterdarstellung:

$$\begin{aligned}
x &= ce^{-z} + 2 - 2z \\
y &= c(1+z)e^{-z} + 2 - z^2
\end{aligned}$$

$$
\begin{aligned}
y &= f(y')x \\
f(z) &= \frac{z^2}{2(z+2)} \\
g(z) &= 0
\end{aligned}
$$

Singuläre Lösung:

$$
\begin{aligned}
\frac{z^2}{2(z+2)} &= z \\
z^2 - 2z^2 - 4z &= 0 \\
z_1 = 0 &\qquad z_2 = 4 \\
y_1 = 0 &\qquad y_2 = -4x
\end{aligned}
$$

Lösungsschar: $f'(z) = \frac{z^2+4z}{2(z+2)^2}$

$$
\begin{aligned}
z - f(z) &= \frac{z^2+4z}{2(z+2)} \\
\frac{\mathrm{d}x}{\mathrm{d}z} &= \frac{f'(z)}{z-f(z)}x = \frac{x}{z+2} \\
x &= c(z+2) \\
y &= \frac{z^2}{2(z+2)}c(z+2) = \frac{c}{z}z^2 = \frac{c}{2}\left(\frac{x}{c}-2\right)^2 = \frac{1}{2c}(x-2c)^2
\end{aligned}
$$

$$
y = \frac{(x-c)^2}{c}
$$

$$
\begin{aligned}
y &= f(y')x + g(y') \\
f(z) &= 1 \\
g(z) &= z^3 - 3z
\end{aligned}
$$

Singuläre Lösung:

$$
\begin{aligned}
f(z) &= z \\
z &= 1 \\
y &= 1x + g(1) \\
y &= x - 2
\end{aligned}
$$

Lösungsschar:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}z} &= \frac{f'(z)}{z-f(z)}x + \frac{g'(z)}{z-f(z)} = \frac{3z^2-3}{z-1} = 3(z+1) \\
x &= \frac{3}{2}z^2 + 3z + c \\
y &= \frac{3}{2}z^2 + c + z^3
\end{aligned}
$$

8. Diferenciální rovnice

$$
y' = g(x)y + h(x)y^\alpha
$$

se nazývá Bernoulliho DR. Proveďte substituci

$$z = y^{1-\alpha}.$$

Jaké lineární diferenciální rovnici $z(x)$ odpovídá?

Řešte Bernoulliho diferenciální rovnici:

$$xy' - 4y = x^2\sqrt{y}$$

$$y^{n-1}(ay' + y) = x, \quad n \in \mathbb{N}, \quad n \neq 0, 1, \quad a \in \mathbb{R}.$$

Eine Differentialgleichung

$$y' = g(x)y + h(x)y^\alpha$$

heisst Bernoullische Differentailgleichung. Wir substituieren

$$z = y^{1-\alpha}$$

Welcher linearen Differentialgliechung für $z(x)$ entspricht die Bernoullische Differentialgleichung?

Lösen Sie die Bernoullische Differentialgleichung

$$xy' - 4y = x^2\sqrt{y}$$

$$y^{n-1}(ay' + y) = x, \quad n \in \mathbb{N}, \quad n \neq 0, 1, \quad a\text{reell}$$

$$\begin{aligned}
Z &= y^{1-\alpha} \\
z' &= \frac{1-\alpha}{y^\alpha}y' \\
y' &= z'\frac{y^\alpha}{1-\alpha} \\
z' &= \frac{1-\alpha}{y^\alpha}(g(x)y + h(x)y^\alpha) \\
z' &= (1-\alpha)g(x)z + (1-\alpha)h(x)
\end{aligned}$$

rechts $y^{\frac{1}{2}=n}$. Substitution $u(x) = y^{1-n} = \sqrt{y}$.

$$\begin{aligned}
u' &= \frac{1}{2\sqrt{y}}y' \\
y' &= 2uu' \\
2xuu' - 4u^2 &= x^2u \\
u' - \frac{2}{x}u &= \frac{x}{2}
\end{aligned}$$

Homogene Lösung:

$$\begin{aligned}
\frac{\mathrm{d}u}{u} &= 2\frac{\mathrm{dx}}{x} \\
u &= cx^2
\end{aligned}$$

Variation der Konstanten:

$$\begin{aligned} x^2 c' &= \frac{x}{2} \\ c' &= \frac{1}{2x} \\ c &= \frac{1}{2}\ln|x| \\ u(x) &= x^2(c + \frac{1}{2}\log|x|) \\ y &= x^4\left(c + \frac{1}{2}\ln|x|\right)^2 \end{aligned}$$

Ansatz: $u = y^n$

$$\begin{aligned} u'' &= ny^{n-1}y' \\ \frac{a}{n}u' + u &= x \\ u' + \frac{n}{a}u &= \frac{n}{a}x \end{aligned}$$

Homogene Lösung: $u(x) = e^{-\frac{n}{a}x}$.
Variation der Konstanten:

$$\begin{aligned} c'(x)e^{-\frac{n}{a}x} &= \frac{n}{a}x \\ c'(x) &= \frac{n}{a}xe^{\frac{n}{a}x} \\ c(x) &= xe^{\frac{n}{a}x} - \int e^{\frac{n}{a}x}\mathrm{d}x = xe^{\frac{a}{n}x} - \frac{a}{n}e^{\frac{n}{a}x} \end{aligned}$$

spezielle inhomogene Lösung:

$$y(x) = x - \frac{a}{n}$$

Allgemeine inhomogene Lösung:

$$y(x) = \frac{c}{e^{\frac{n}{a}x}} + x - \frac{a}{n}$$