2019 - Početní část 1

implicitní funkce

③ 6b Ověřte, že vztahy $u^2v - \cos x + \ln v = 0$ ← F_1

$\ln u + x^3 - xv^2 - \sin(uvx) = 0$ ← F_2

zaručují na nějakém okolí bodu $u = v = 1,\ x = 0$ $\exists$ fci $u = u(x)$, $v = v(x)$ + spočtěte $u'(0)$ a $v'(0)$.

① $F_1(0,1,1) = 1 - 1 + 0 = 0$ $\quad F_2(0,1,1) = 0 + 0 + 0 - 0 = 0$ ✓

② $\dfrac{\partial F_1}{\partial u} = 2uv \qquad \dfrac{\partial F_1}{\partial v} = u^2 + \dfrac{1}{v}$

$\dfrac{\partial F_2}{\partial u} = \dfrac{1}{u} - \cos(uvx)\cdot vx \qquad \dfrac{\partial F_2}{\partial v} = -2xv - \cos(uvx)\cdot ux$

$\rightarrow \det\begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix} = -2 \neq 0$ ✓

③ fce je třídy min C^1 ✓

Dále: $\dfrac{\partial F_1}{\partial x} = 2uv\dfrac{\partial u}{\partial x} + u^2\dfrac{\partial v}{\partial x} + \sin x + \dfrac{1}{v}\dfrac{\partial v}{\partial x}$

$\Rightarrow 2\dfrac{\partial u}{\partial x} + 2\dfrac{\partial v}{\partial x} = 0 \rightarrow \dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial x}$

$\dfrac{\partial F_2}{\partial x} = \dfrac{1}{u}\dfrac{\partial u}{\partial x} + \cancel{3x^2} - v^2 - \cancel{2xv\dfrac{\partial v}{\partial x}} - \cos(uvx)\left(uv + \cancel{\dfrac{\partial u}{\partial x}vx} + \cancel{\dfrac{\partial v}{\partial x}ux}\right)$

$= \dfrac{\partial u}{\partial x} - 1 - 1 \rightarrow \dfrac{\partial u}{\partial x} = 2 \rightarrow \dfrac{\partial v}{\partial x} = -2$

$\nabla u(0) = 2 \qquad \nabla v(0) = -2$

2019 - Početní část 6

Složitější implicitní funkce + návod

(3) [7b] Spočtěte $\frac{\partial z}{\partial x}$ a $\frac{\partial z}{\partial y}$ v bodě $u=2, v=1$, kde

$x = u + v^2 \quad y = u^2 - v^2 \quad z = 2uv$ + Návod (viz. opravdové řešení)

Ověření dle věty o impl. fci:

① $x = 2+1 = 3 \quad y = 4-1 = 3 \quad z = 4$ ✓

② $\frac{\partial x}{\partial u} = 1 \quad \frac{\partial x}{\partial v} = 2v \quad \frac{\partial y}{\partial u} = 2u \quad \frac{\partial y}{\partial v} = -2v \quad \frac{\partial z}{\partial u} = 2v \quad \frac{\partial z}{\partial v} = 2u$

pro $u = u(x,y)$, $v = v(x,y)$

$\begin{pmatrix} 1 & 2 \\ \cancel{2}\,4 & -2 \end{pmatrix} = -2 - \cancel{8} = \cancel{-6}\,10 \neq 0$ ✓

③ Fce je hladká ✓

Dále $\frac{\partial x}{\partial x} = \frac{\partial}{\partial x}\, u + v^2 = \frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = \cancel{0}\;1$

$\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$

nezapomeň!

$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\, u^2 - v^2 = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x} = 0$

$2u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} = \cancel{0}\;1$

$\cancel{\frac{\partial u}{\partial x} = -2v\frac{\partial v}{\partial x} = -2\frac{\partial v}{\partial x}}$

$\cancel{-8\frac{\partial v}{\partial x} - 2\frac{\partial v}{\partial x} = 0}$

$\cancel{\rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0}$

$\frac{\partial u}{\partial x} = 1 - 2\frac{\partial v}{\partial x}$

$\rightarrow 4 - 8\frac{\partial v}{\partial x} - 2\frac{\partial v}{\partial x} = 0$

$10\frac{\partial v}{\partial x} = 4 \rightarrow \frac{\partial v}{\partial x} = \frac{2}{5}$

$\rightarrow \frac{\partial u}{\partial x} = \frac{1}{5}$

$4\frac{\partial u}{\partial y} = 1 + 2\frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = \frac{1}{4} + \frac{1}{2}\frac{\partial v}{\partial y}$

$\frac{1}{4} + \frac{1}{2}\frac{\partial v}{\partial y} + 2\frac{\partial v}{\partial y} = 0$

$\cancel{\frac{5}{4}}\;\frac{10}{4}\frac{\partial v}{\partial y} = -\frac{1}{4}$

$\rightarrow \frac{\partial v}{\partial y} = -\frac{1}{\cancel{3}\,10}$

$\rightarrow \frac{\partial u}{\partial y} = \cancel{\frac{2}{3}}\;\frac{1}{5}$

$\Rightarrow \frac{\partial z}{\partial x} = 2v\frac{\partial u}{\partial x} + 2u\frac{\partial v}{\partial x} = \frac{2}{5} + \frac{8}{5} = 2$

$\frac{\partial z}{\partial y} = 2\frac{\partial u}{\partial y} + 4\frac{\partial v}{\partial y} = \frac{2}{5} - \frac{4}{10} = 0$

2019 - Početní část 2

Vyšetřit spojitost, parc. der., tot. dif. závislost na α

③ 6b V závislosti na $\alpha \in \mathbb{R}$ vyšetřete, zda fce $f(x,y) = \begin{cases} \dfrac{x^2y^3}{(x^2+y^2)^\alpha} & ,(x,y) \in \mathbb{R}^2 \setminus (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$

je v počátku ⓐ spojitá

ⓒ má totální diferenciál

ⓑ má parc. derivace

ⓐ $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{(x^2+y^2)^\alpha} \overset{\text{polární substituce}}{=} \lim_{r\to 0^+} \frac{r^5 \sin^2\varphi \cos^3\varphi}{r^{2\alpha}} = \lim \underbrace{r^{5-2\alpha}}_{\text{musí jít k nule}} \cdot$ (omezené renlové fce)

splněno pro $\alpha < \frac{5}{2}$

ⓑ řeší z definice:

pro x: $\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \frac{\lim_{h\to 0} \frac{h^2 \cdot 0^3}{h^{2\alpha}}}{h} = \lim_{h\to 0} h^{2-2\alpha-1}$ $= \lim_{h\to 0} h^{1-2\alpha}$

$\to (h,0)$

~~splněno pro $\alpha < \frac{1}{2}$~~

pro y: $\lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} h^{2-2\alpha}$ ~~splněno pro $\alpha < 1$~~

$\to (0,h)$

Postup byl dobře, jen jsem zapomenul uvažovat to $y^3 = 0^3 \to h^2 \cdot 0^3 = 0$ a analogicky pro $\frac{\partial f}{\partial y}$.

Tedy $\frac{0-0}{h} = 0 \to$ splněno pro $\forall \alpha \in \mathbb{R}$

ⓒ Tot. dif.?

$$\lim_{h\to 0} \frac{f(h_1,h_2) - f(0,0) - Lh}{\|h\|} \overset{?}{=} 0$$

$$= \lim_{h\to 0} \frac{\frac{h_1^2 h_2^3}{(h_1^2+h_2^2)^\alpha}}{\|h\|}$$

nechť $\|h\|$ je $\sqrt{h_1^2+h_2^2}$ $\to (\sqrt{h_1^2+h_2^2})^{2\alpha}$

$$= \lim_{h\to 0} \frac{\|h\|^5}{\|h\|^{2\alpha+1}} = \lim_{h\to 0} \|h\|^{4-2\alpha}$$ splněno pro $\alpha < 2$

~~$\frac{\partial f(x,y)}{\partial x} = \frac{2xy^3(x^2+y^2)^\alpha - x^2y^3\alpha(x^2+y^2)^{\alpha-1}2x}{(x^2+y^2)^{2\alpha}}$~~

2019 - Početní část 4

Rovnice ve tvaru tot. dif + integrační faktor

(3) [7b] Ověřte, že rovnice $y^3+xy^2+x^2y+y+2x+(xy^2+x^2y+x^3+x+2y)\frac{\partial y}{\partial x}=0$ není ve tvaru totálního diferenciálu. Metodou int. faktoru $(m=\Phi(xy))$ nalezněte její obecné řešení.

$M = y^3+xy^2+x^2y+y+2x \qquad N = xy^2+x^2y+x^3+x+2y$

$\frac{\partial}{\partial y}M \overset{?}{=} \frac{\partial}{\partial x}N$

$\frac{\partial}{\partial y}M = 3y^2+2xy+x^2+1$

$\frac{\partial}{\partial x}N = y^2+2xy+3x^2+1$

liší se.

$$\boxed{\frac{m'}{m} = \frac{\frac{\partial}{\partial x}N - \frac{\partial}{\partial y}M}{M\frac{\partial}{\partial y}\Phi - N\frac{\partial}{\partial x}\Phi}} \qquad \frac{\partial}{\partial y}\Phi = x \qquad \frac{\partial}{\partial x}\Phi = y$$

Per Partes:
$(f\cdot g)' = f'g + g'f$
$f'g = -fg' + (f\cdot g)'$
$\int f'g\,dx = -\int fg'\,dx + fg$
$f \dots e^{xy}$

$$\frac{m'}{m} = \frac{y^2+\cancel{2xy}+3x^2+\cancel{1}-3y^2-\cancel{2xy}-x^2-\cancel{1}}{\cancel{xy^3}+\cancel{x^2y^2}+\cancel{x^3y}+xy+2x^2-\cancel{xy^3}-\cancel{x^2y^2}-\cancel{yx^3}-\cancel{xy}-2y^2} = \frac{2x^2-2y^2}{2x^2-2y^2} = 1$$

$\longrightarrow \quad m = e^{\int 1\,dz} = e^z = \boxed{e^{xy}}$ ← integrační faktor

$\int Me^{xy}dx = \int y^3e^{xy}+xy^2e^{xy}+x^2ye^{xy}+ye^{xy}+2xe^{xy}\,dx$
$= e^{xy}y^2 + e^{xy} + \int xy^2e^{xy}dx + \int x^2ye^{xy}dx + \int 2xe^{xy}dx$
$= e^{xy}y^2 + e^{xy} + xye^{xy} - \int y^2e^{xy}dx + x^2e^{xy} - 2\int xye^{xy}dx + 2\frac{x}{y}e^{xy}$
$= e^{xy}y^2 + e^{xy} + xye^{xy} - ye^{xy} + x^2e^{xy} - 2xe^{xy} + 2e^{xy} - \int 2e^{xy}\dots$
$+ 2\frac{x}{y}e^{xy} - 2\frac{1}{y}e^{xy}$

NAVIC SE PER PARTES!

$\frac{\partial}{\partial y}e^{xy}(y^2+xy+x^2)+C_{(y)}$
$= e^{xy}\left[2y+y+x(y^2+xy+x^2)\right]+C$
↳ sedí s tou druhou částí rovnice.

$\Rightarrow$ Znovu: $\int e^{xy}(y^3+xy^2+x^2y+y+2x)\,dx$
$= y^2e^{xy}+xye^{xy}-\int ye^{xy}dx+x^2e^{xy}-2\int xe^{xy}dx+e^{xy}$
$+2\frac{x}{y}e^{xy}-\int 2\frac{1}{y}e^{xy}$
$= e^{xy}\left(y^2+xy+x^2+\cancel{1}+\cancel{2\frac{x}{y}}\right)+e^{xy}\left(-\cancel{1}-\cancel{2\frac{1}{y^2}}-\cancel{2\frac{x}{y}}\right)+e^{xy}\left(\cancel{2\frac{1}{y^2}}\right)$
$= e^{xy}(y^2+xy+x^2)+C(y)$ → Normálně bych ještě analogicky vypočítal tu druhou část, a pak to porovnal, ale stačí se podívat: (a udělat to takle)

$\Rightarrow$ Řešení $F(x,y) = e^{xy}(y^2+xy+x^2)$

implicitní funkce

2019 – Početní část 3

③ 7b Ověřte, že vztahy
$$F_1:\ e^{\frac{1}{uv}-1} - \ln(ux+vy+e) = 0$$
$$F_2:\ e^{\frac{-1}{uv^2}-1} - \ln(uy+vx+e) = 0$$
zaručují na nějakém okolí bodu $x=y=0$, $u=v=1$ $\exists$ fcí $u=u(x,y)$, $v=v(x,y)$ + spočtěte $\nabla u(0,0)$ a $\nabla v(0,0)$

① $F_1(0,0,1,1) = e^0 - \ln(0+0+e) = 1-1 = 0$

$F_2(0,0,1,1) = e^0 - \ln(e) = 0$ ✓

② $$\frac{\partial F_1}{\partial u} = -e^{\left(\frac{1}{uv}-1\right)}\cdot\frac{1}{u^2v} - \frac{1}{ux+vy+e}\cdot x$$
$$\frac{\partial F_1}{\partial v} = -e^{\left(\frac{1}{uv}-1\right)}\cdot\frac{1}{v^2u} - \frac{1}{ux+vy+e}\cdot y$$
$$\frac{\partial F_2}{\partial u} = e^{\left(\frac{-1}{uv^2}-1\right)}\frac{1}{u^2v^2} - \frac{1}{uy+vx+e}\cdot y$$
$$\frac{\partial F_2}{\partial v} = 2e^{\left(\frac{-1}{uv^2}-1\right)}\cdot\frac{1}{uv^3} - \frac{1}{uy+vx+e}\cdot x$$

$$\det\begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = -1+2 = 1 \neq 0 \ ✓$$

③ okolo bodu $(0,0,1,1)$ je fce třídy C^1 (resp. hladké). (možná i C^k, ale toto stačí, rychleji řešit silnější tvrzení bych ztratil body)

$$\frac{\partial f_1}{\partial x} = -e^{\left(\frac{1}{uv}-1\right)}\left(\frac{1}{u^2v}\frac{\partial u}{\partial x} + \frac{1}{uv^2}\frac{\partial v}{\partial x}\right) - \left(\frac{1}{ux+vy+e}\right)\cdot\left(\frac{\partial u}{\partial x}\cdot x + u + \frac{\partial v}{\partial x}y\right)$$
$$= -e^0\cdot\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right) - \frac{1}{e} \to \frac{\partial u}{\partial x}+\frac{\partial v}{\partial x} = -\frac{1}{e}$$

$\frac{\partial f_1}{\partial y}$: $\Rightarrow \frac{\partial u}{\partial y}+\frac{\partial v}{\partial y} = -\frac{1}{e}$ ze symetrie

$$\frac{\partial f_2}{\partial x}: \stackrel{\sim}{=} e^0\cdot\left(\frac{\partial u}{\partial x}\cdot 1 + \frac{\partial v}{\partial x}\cdot 2\right) - \frac{1}{e} \Rightarrow \frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = \frac{1}{e}$$

$$\frac{\partial f_2}{\partial y}: \to \frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = \frac{1}{e}$$

$$\frac{\partial u}{\partial x} = -\frac{1}{e} - \frac{\partial v}{\partial x}$$
$$-\frac{1}{e} + \frac{\partial v}{\partial x} = \frac{1}{e}$$
$$\frac{\partial v}{\partial x} = +\frac{2}{e}$$
$$\to \frac{\partial u}{\partial x} = \frac{\cancel{1}\,-3}{e}$$

analogicky pro y:
$$\frac{\partial v}{\partial y} = +\frac{2}{e}$$
$$\frac{\partial u}{\partial y} = \frac{\cancel{1}\,-3}{e}$$

$$\nabla v(0,0) = \left(+\frac{2}{e}, +\frac{2}{e}\right)$$
$$\nabla u(0,0) = \left(\frac{\cancel{1}\,-3}{e}, \frac{\cancel{1}\,-3}{e}\right)$$

2019 - Početní část 7

Implicitní funkce

③ 6b Ověřte, že vztahy $x^3 + xy + \sin((u+v)\pi) - e^{uv} = 1$ F_1

$x^3 - xy + \cos((u+v)\pi) - e^{u^2v^2} + u + v = 2$ F_2

zaručují na nějakém okolí bodu

$u = 0 \quad v = x = y = 1 \quad \to (x, y, u, v)$ je $(1, 1, 0, 1)$

$\exists$ fcí $u = u(x, y)$ a $v = v(x, y)$ + spočtěte $\nabla u(1,1)$ a $\nabla v(1,1)$

① Funkční hodnota těch fcí v tom bodě:

$F_1(1,1,0,1) = 1 + 1 + 0 - 1 = 1$

$F_2(1,1,0,1) = 1 - 1 + (-1) - 1 + 1 = -1$

②
$$^1\ \frac{\partial F_1}{\partial u} = \pi \cdot \cos((u+v)\pi) - ve^{uv}$$
$$^2\ \frac{\partial F_1}{\partial v} = \pi \cdot \cos((u+v)\pi) - ue^{uv}$$
$$^3\ \frac{\partial F_2}{\partial u} = \pi \cdot (-1) \cdot \sin((u+v)\pi) - 2uv^2 e^{u^2v^2} + 1$$
$$^4\ \frac{\partial F_2}{\partial v} = \pi(-1) \cdot \sin((u+v)\pi) - 2vu^2 e^{u^2v^2} + 1$$

$$\det\begin{pmatrix} -\pi - 1^{\,1} & -\pi^{\,2} \\ 1^{\,3} & 1^{\,4} \end{pmatrix} = -\pi - 1 + \pi = -1 \neq 0$$

③ Fce je hladká na def. oboru

$\Rightarrow$ splněny předpoklady $\exists$ $u = u(x,y)$, $v = v(x,y)$

Dále: F_1 a F_2 zderivujeme zvlášť podle x a y. Uvědomme si, že u a v jsou funkce. To abychom našli ten gradient.

$$\frac{\partial F_1}{\partial x} \to 3x^2 + y + \pi\frac{\partial u}{\partial x} \cdot \cos((u+v)\pi) + \pi\frac{\partial v}{\partial x} \cdot \cos((u+v)\pi) - \frac{\partial u}{\partial x} \cdot ve^{uv} - \frac{\partial v}{\partial x} ue^{uv} = 0$$

$$\frac{\partial F_2}{\partial x} \to 3x^2 - y - \sin((u+v)\pi)\pi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) - e^{u^2v^2}\left(\frac{\partial u}{\partial x} 2uv^2 + \frac{\partial v}{\partial x} 2vu^2\right) + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial F_1}{\partial y} \to x + \pi\cos((u+v)\pi)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y} \cdot ve^{uv} - \frac{\partial v}{\partial y} ue^{uv} = 0$$

$$\frac{\partial F_2}{\partial y} \to -||- \text{ analogicky.}$$

$$3 + 1 - \pi\frac{\partial u}{\partial x} - \pi\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = 0 \to 4 = \pi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial x} \quad \Big\{ 4 = \pi \cdot 2 + \frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2\pi + 4$$

$$3 - 1 + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \to \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = -2 \quad \frac{\partial v}{\partial x} = -2 - \frac{\partial u}{\partial x} \quad \to \frac{\partial v}{\partial x} = -2\pi - 6$$

$$1 - \pi\frac{\partial u}{\partial y} - \pi\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} = 0 \longrightarrow 1 - \cancel{\pi\frac{\partial u}{\partial y}} - \pi + \cancel{\pi\frac{\partial u}{\partial y}} - \frac{\partial u}{\partial y} = 0$$

$$-1 + \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 \to \frac{\partial v}{\partial y} = 1 - \frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial y} = 1 - \pi$$

$$\to \frac{\partial v}{\partial y} = \pi$$

$$\Longrightarrow \nabla u(1,1) = (2\pi + 4, 1 - \pi) \quad \nabla v(1,1) = (-2\pi - 6, \pi)$$

2019 - Početní část 1 Extrémy na $\mathbb{R}$

(4) 7b Nalezněte všechny lok. extrémy fce (+ jaké extrémy to jsou)

$f(x,y) = 3x + 2y + \ln(x^2+y^2) - \operatorname{arctg}\left(\frac{x}{y}\right)$ na množině $\mathbb{R}^2 \setminus \{(x,0); x \in \mathbb{R}\}$.

$$\frac{\partial f}{\partial x} = 3 + \frac{1}{x^2+y^2}\cdot 2x - \frac{1}{1+\frac{x^2}{y^2}}\cdot\frac{1}{y} = 3 + \frac{2x}{x^2+y^2} - \frac{y}{y^2+x^2} = 3 + \frac{2x-y}{y^2+x^2} = \frac{3y^2+3x^2+2x-y}{y^2+x^2}$$

$$\frac{\partial f}{\partial y} = 2 + \frac{1}{x^2+y^2}\cdot 2y - \frac{1}{1+\frac{x^2}{y^2}}\cdot(-1)\frac{x}{y^2} = 2 + \frac{2y}{x^2+y^2} + \frac{y^2}{y^2+x^2}\frac{x}{y^2} = 2 + \frac{2y+x}{x^2+y^2}$$

$$= \frac{2x^2+2y^2+2y+x}{x^2+y^2}$$

$$\cancel{3 + \frac{2x-y}{y^2+x^2} - 2 + \frac{2y+x}{y^2+x^2} = 0}$$

$$\cancel{1 = \frac{2y+x-2x+y}{y^2+x^2} = \frac{3y-x}{y^2+x^2}}$$

$$\cancel{y^2+x^2 = 3y-x}$$

$$\cancel{0 = x^2+y^2+x-3y}$$

$$\cancel{0 = x(x+1) + y(y-3)}$$

slepé ulička

$$\cancel{2x^2+2y^2+2y+x = 3y^2+3y^2+2x-y}$$

$$\cancel{-x+3y = x^2+y^2}$$

slepé ulička

$$\begin{cases} 3(x^2+y^2)+2x-y=0 & /\cdot -2 \\ 2(x^2+y^2)+2y+x=0 & /\cdot 3 \end{cases}$$

$$-4x+2y+6y+3x=0$$

$$\underline{8y = x}$$

✓

$$\to 0 = 2 + \frac{10y}{64y^2+y^2} \to -2\cdot y^2 = \frac{1}{6{,}5}y$$

$$-13{,}0\,y^2 = y$$

$$y\cdot(1+13y) = 0$$

$y = 0$ nebo $y = -\frac{1}{13}$ $\to$ $x = -\frac{8}{13}$

NE 1 podm.

$$\frac{\partial^2 f}{\partial x^2} = \frac{2(y^2+x^2) - 2x(2x-y)}{(y^2+x^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2}\bigg|_{(\,,\,)} = \frac{2\left(\frac{1}{13^2}+\frac{64}{13^2}\right)+\frac{16}{13}\left(\frac{-16}{13}+\frac{1}{13}\right)}{\left(\frac{65}{13^2}\right)^2}$$

$$= \cancel{\frac{130-15\cdot 16}{13^2}\cdot\frac{13^2}{65}} \quad \cancel{\frac{130-240}{65}}$$

$$\to 2\left(\frac{1}{13^2}+\frac{64}{13^2}\right)+\frac{16}{13}\left(\frac{-15}{13}\right) = \frac{130-240}{13^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2(y^2+x^2) - 2y(2y+x)}{(y^2+x^2)^2}$$

$$\to 2\cdot(1+64)+2(-17) = +$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{(x^2+y^2)+2x(2y+x)}{(x^2+y^2)^2}$$

$$\to (65) - 16(-16-1) = -$$

$$\to \begin{pmatrix} - & - \\ - & + \end{pmatrix} \to \text{sedlový bod}$$

OVĚŘIT POŘÁDNĚJI! Sylvester.

✓

Extrémy s vazbou

2019 – Početní část 2

(4) 9b Nalezni max. a min. hodnotu fce $f(x,y,z) = x^3 + y^3 + z^3$

na množině $\frac{x^2}{2^2} + \frac{y^2}{3^2} + z^2 \leq 1$

VNITŘEK $\vec{\nabla} f = (3x^2, 3y^2, 3z^2)$ $\longrightarrow$ jediné možné min. v $(0,0,0)$, to ale není nejmenší hodnota, např. v $(-1,0,0)$ je funkční hodnota nižší.

Max. a min. se tedy nabývá na okraji

OKRAJ $F(x,y,z,\lambda) = x^3 + y^3 + z^3 - \lambda\left(\frac{x^2}{2^2} + \frac{y^2}{3^2} + z^2 - 1\right)$

$\frac{\partial F}{\partial x} = 3x^2 - \lambda \frac{x}{2}$ $\qquad x\left(3x - \frac{\lambda}{2}\right) = 0 \longrightarrow \lambda = 6x \rightarrow x^2 = \frac{\lambda^2}{36} \rightarrow 3^2 \cdot 2^2$

$\frac{\partial F}{\partial y} = 3y^2 - \lambda \frac{2}{9} y$ $\qquad y\left(3y - \frac{2}{9}\lambda\right) = 0$ $\qquad \lambda = \frac{27}{2} y \rightarrow y^2 = \frac{4}{3^6}\lambda^2$

$\frac{\partial F}{\partial z} = 3z^2 - \lambda 2z$ $\qquad z(3z - 2\lambda) = 0$ $\qquad \lambda = \frac{3}{2} z \rightarrow z^2 = \frac{4}{9}\lambda^2 \rightarrow 3^2$

$\frac{x^2}{2^2} + \frac{y^2}{3^2} + z^2 = 1 \longrightarrow \lambda^2\left(\frac{1}{2^4 \cdot 3^2} + \frac{2^2}{3^8} + \frac{2^2}{3^2}\right) = 1$

$\lambda^2\left[\frac{1}{2^4 \cdot 3^8} \cdot \left(3^6 + 2^6 + 3^6 \cdot 2^6\right)\right] = 1$

ooo zbytečně složité

$x = 0$ nebo $x = \frac{\lambda}{6}$

$y = 0$ nebo $y = \frac{2}{27}\lambda$

$z = 0$ nebo $z = \frac{2}{3}\lambda$

Jak řešit? Proskoumat všechny superpozice a zjistit jaké je největší/nejmenší.

i) $x = y = z = 0$ ... není, dokázáno výše

ii) $x = 0 \quad y = 0 \quad z = 1 \qquad f(0,0,1) = 1$

$x = 0 \quad y = 3 \quad z = 0 \qquad f(0,3,0) = 27$

$x = 2 \quad y = 0 \quad z = 0 \qquad f(2,0,0) = 8$

iii) $x = \frac{\lambda}{6} \quad y = \frac{2}{27}\lambda \quad z = 0 \qquad \frac{\lambda^2}{4 \cdot 36} + \frac{\lambda^2 \cdot 4}{3^8} = 1$ ... nalezneme λ, určíme x a y, vypočítáme.

Ez a rozumím tomu, ale numericky nerealizovatelný krok.

Max v $(0,3,0)$ min v $(0,-3,0)$

2019 - Početní část 4

Extrémy s vazbou

4 [9b] Max a min hodnoty + kde má fce extrémy.

$$f(x,y) = |x| + |y| \ , \text{ vazba } \frac{x^2}{2^2} + \frac{y^2}{3^2} \leq 1$$

VNITŘEK

$$\nabla f = (\text{sign}(x), \text{sign}(y)) \longrightarrow$$ pro $(0,0)$ vyhovuje nutné podmínce pro $\exists$ extrému. Vyhovuje vazební podmínce. Díky abs. hodnotám nejnižší hodnota

$$\text{min. } f(0,0) = 0$$

HRANICE

$$F(x,y,\lambda) = |x| + |y| - \lambda\left(\frac{x^2}{2^2} + \frac{y^2}{3^2} - 1\right)$$

dosadíme

$$0 = \text{sign}|x| - \lambda \frac{x}{2} \longrightarrow x = \frac{2}{\lambda} \text{sign}|x|$$

$$0 = \text{sign}|y| - \lambda \frac{2}{3^2} y \qquad y = \frac{9}{2\lambda} \text{sign}|y|$$

$$1 = \frac{x^2}{2^2} + \frac{y^2}{3^2} \longrightarrow$$

$$1 = \frac{2^2}{2^2} \frac{1}{\lambda^2} + \frac{3^4}{3^2 2^2} \frac{1}{\lambda^2}$$

$$1 = \frac{1}{\lambda^2}\left(1 + \frac{9}{4}\right) = \frac{1}{\lambda^2} \frac{13}{4}$$

$$\lambda^2 = \frac{13}{4} \longrightarrow \lambda = \frac{\sqrt{13}}{2}$$

$$x = \pm \frac{4}{\sqrt{13}}$$

$$y = \pm \frac{9}{\sqrt{13}}$$

$\longrightarrow$ kombinace těchto bodů

$-$ Maximum $f\left(\pm \frac{4}{\sqrt{13}}, \pm \frac{9}{\sqrt{13}}\right) = \sqrt{13}$

+ ještě kombinace sign(x) nebo sign(y) = 0

LO7A - Početní část 3

Extrémy na R

→ lok. extrémy

(4) 7b

$$f(x,y) = (2x+2y+3)e^{-(x^2+y^2)}$$ definována na $\mathbb{R}^2$

$$\frac{\partial f}{\partial x} = e^{-x^2-y^2}\cdot(2-2x)(\quad)$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2}\cdot(2-2y)(\quad)$$

chyba v derivaci příliš v/c přepůlky

→ podezřelý bod: $x=y=1$ $\quad$ $(1 \quad 2-2x = 2-2y) = 0$

$$\frac{\partial^2 f}{\partial x^2} = e^{-x^2-y^2}\cdot(-2-2x(2-2x)) = e^{..}\cdot(4x^2-4x-2)$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-x^2-y^2}\cdot(-2-2y(2-2y)) = e^{..}\cdot(2y^2-2y-1)\cdot 2$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = -2y\cdot e^{-y^2-x^2}(2-2x) = -4y\,e^{-y^2-x^2}(1-x)$$

analogicky

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(1,1)} = e^{-2}\cdot(-2-2\cdot 0) = -2e^{-2}$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(1,1)} = -2e^{-2} \longrightarrow \begin{pmatrix} - & 0 \\ 0 & - \end{pmatrix}$$

$$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(1,1)} = 0$$

v (1,1) lok max

DRUHÁ STRANA SPRÁVNĚ!

ZNOVU

$$\frac{\partial}{\partial x}(2x+2y+3)e^{-x^2-y^2} = 2e^{-x^2-y^2} - 2x(2x+2y+3)e^{-x^2-y^2}$$

$$\frac{\partial f}{\partial y} = e^{-x^2-y^2}\Big(2-2y(2x+2y+3)\Big)$$

$$\ldots \overset{!}{=} 0$$

$$2-2y(2x+2y+3) = 2-2x(2y+2x+3) \overset{!}{=} 0$$

→ • $y = x = 0$ $\quad$ ↳ $y = x$

• $y = 0 \quad x \neq 0 \rightarrow x = -\frac{3}{2}$ $\quad$ ↳ $2-2x(2x+2x+3) = 0$

• $x = 0 \quad y = -\frac{3}{2}$ $\quad$ $2-8x^2-6x = 0$

$\quad 4x^2+3x-1 = 0$

• $x \neq 0 \quad y \neq 0 \longrightarrow (2x+2y+3) = 0$ $\quad$ $(4x-1)(x+1) = 0$

podezřelé body $(\frac{1}{4}, \frac{1}{4})$, $(-1,-1)$ $\quad$ $x_1 = \frac{1}{4} = y_1 \quad x_2 = -1 = y_2$

$$\frac{\partial^2 f}{\partial x^2} = -2x e^{-x^2-y^2}\cdot\big(2-2x(2x+2y+3)\big) + e^{-x^2-y^2}\cdot(-8x-4y-6)$$

$$= e^{-x^2-y^2}\Big[-2x\big(2-2x(2x+2y+3)\big) - 8x - 4y - 6\Big]$$

... díky $x = y$ není třeba počítat $\frac{\partial^2 f}{\partial y^2}$

$$\hookrightarrow v(-1,-1) = 2\cdot\underbrace{\big(2+2\overbrace{(-2-2+3)}^{-1}\big)}_{0} + 8 + 4 - 6 = 6$$

$$v(\tfrac{1}{4},\tfrac{1}{4}) = -\frac{1}{2}\underbrace{\big(2-\tfrac{1}{2}(\tfrac{1}{2}+\tfrac{1}{2}+3)\big)}_{0} - 2 - 1 - 6 = -$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{-x^2-y^2}\Big[(-4y) - 2x\big(2-2y(2x+2y+3)\big)\Big]$$

$$v(-1,-1) = 4 + 2\underbrace{(2 - \ldots)}_{0} = 4$$

$$v(\tfrac{1}{4},\tfrac{1}{4}) = -1$$

pro $(-1,-1)$ ... $\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$ ... lok min

$(\frac{1}{4}, \frac{1}{4})$ ... $\begin{pmatrix} -9 & -1 \\ -1 & -9 \end{pmatrix}$ ... lok max .... SEDÍ TO!

Extrémy na $\mathbb{R}$

④ (a) b Nalezněte lokální a globální extrémy fce

$$f(x,y) = xy\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$ na jejím definičním oboru. $a,b>0$

podm. $+\frac{x^2}{a^2}+\frac{y^2}{b^2}\leq 1$

$$\frac{\partial f}{\partial x} = y\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} + \left(\frac{-2x}{a^2}\right)\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\cdot x\cdot\frac{1}{2}\cdot y$$

$$= y\cdot\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}} - \frac{x^2y}{a^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}} \stackrel{!}{=} 0 \quad / \cdot\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$\frac{\partial f}{\partial y} = x\cdot\left(1-\frac{y^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}} - \frac{y^2x}{b^2}\underbrace{\left(1-\frac{x^2}{a^2}-\frac{y}{b^2}\right)^{-\frac{1}{2}}}_{\text{nesmí být } 0} \stackrel{!}{=} 0 \quad / \cdot\sqrt{}$$

$$x\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-\frac{y^2x}{b^2} = 0 = x\left(1-\frac{x^2}{a^2}-2\frac{y^2}{b^2}\right)$$

$$0 = y\left(1-2\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

$\longrightarrow$ $x=y=0$

$x=0 \quad y=|b|$

$y=6 \quad x=|a|$

$$-2\frac{x^2}{a^2}-\frac{y^2}{b^2} = -\frac{x^2}{a^2}-2\frac{y^2}{b^2}$$

$$2\frac{x^2}{a^2}-\frac{x^2}{a^2} = 2\frac{y^2}{b^2}-\frac{y^2}{b^2}$$

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{1}{3} \quad x=\sqrt{\frac{1}{3}}a$$

$$y=\sqrt{\frac{1}{3}}b$$

ooo

2019 – Početní část G

Extrémy na R

(4) 8b Nalezněte lokální extrémy funkce $f(x,y) = (5x+7y-25)e^{-(x^2+xy+y^2)}$ na množině $\mathbb{R}^2$. Jaké extrémy to jsou?

$$\frac{\partial f}{\partial x} = 5e^{-(x^2+xy+y^2)} + (5x+7y-25)(-2x-y)\,e^{-(x^2+xy+y^2)}$$

$$= e^{-(x^2+xy+y^2)}\left[5 - 10x^2 - 5xy - 14xy - 7y^2 + 50x + 25y\right)$$

$$= e^{-(x^2+xy+y^2)}\left(-7y^2 - 10x^2 - 19xy + 50x + 25y + 5\right)$$

asi zbytečné

$$\cancel{= -e^{-(x^2+xy+y^2)}\left(10x(x-5)\right)}$$

$$= e^{-(x^2+xy+y^2)}\left(5 - (2x+y)(5x+7y-25)\right)$$

$$\frac{\partial f}{\partial y} = e^{-(x^2+xy+y^2)}\left(7 - (2y+x)(5x+7y-25)\right)$$

$(\ldots) = 0$?

$5 = (2x+y)(5x+7y-25)$

$\frac{5}{2x+y} = 5x+7y-25$

$(\ldots) = 0$?

$\frac{7}{2y+x} = 5x+7y-25$

$\rightarrow \frac{7}{2y+x} = \frac{5}{2x+y} \rightarrow 10y+5x = 14x+7y$

$3y = 9x$

$\boxed{y = 3x}$

$5 - 5x(5x+21x-25) = 0$

$1 - (26x^2 + 25x) = 0$

$-(26x^2+25x-1) = 0$

$(x-1)(-26x-1) = 0$

$x_1 = 1 \quad y_1 = 3$

$x_2 = -\frac{1}{26} \quad y_2 = -\frac{3}{26}$

dále najdu pro tři druhé parciální derivace $\left(\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}\right)$, sestrojím Hessovu matici tak, že do ní dosadím hodnoty z podezřelých bodů.

Pokud je matice pozdef. – lok min

neg def – lok max

smíšené – sedlový bod

000 Nedopočítáno

2019 - Početní část 7.

(4)[86] $f(x,y,z) = |x|+|y|+|z|$ na podmnož. $\mathbb{R}^3$ splňující $x^2 + \frac{y^2}{4} + \frac{z^2}{16} \leq 1$

→ nalezni max a min. hodnotu funkce

Vnitřek: $\nabla f = (\operatorname{sign}|x|, \operatorname{sign}|y|, \operatorname{sign}|z|)$

minimum v $f(0,0,0) = 0$ ↗ nic není nějakého

jinde není splněna nutná podm. ∃ extrému

+ předpoklady: je spojitá, $M = \{(x,y,z) \in \mathbb{R}^3 : x^2 + \frac{y^2}{4} + \frac{z^2}{16}\}$ je kompakt

⇒ min a max existují.

Hranice:
$$F(x,y,z,\lambda) = |x|+|y|+|z| - \lambda\left(x^2 + \frac{y^2}{4} + \frac{z^2}{16} - 1\right)$$

$$\rightarrow \quad \begin{aligned} 0 &= \operatorname{sign}|x| - 2\lambda x \\ 0 &= \operatorname{sign}|y| - \tfrac{1}{2}\lambda y \\ 0 &= \operatorname{sign}|z| - \tfrac{1}{8}\lambda z \\ 1 &= x^2 + \frac{y^2}{4} + \frac{z^2}{16} \end{aligned} \quad \longrightarrow \quad \begin{aligned} x &= \operatorname{sign}|x| \frac{1}{2\lambda} \\ y &= \operatorname{sign}|y| \frac{2}{\lambda} \\ z &= \operatorname{sign}|z| \frac{8}{\lambda} \end{aligned}$$

$$1 = \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{4}{\lambda^2}$$

$$\lambda^2 = \frac{1}{4} + 1 + 4 = \frac{1+4+16}{4} = \frac{21}{4}$$

$$\lambda = \pm\frac{\sqrt{21}}{2}$$

$$x = \pm\frac{1}{\sqrt{21}}, \quad y = \pm\frac{4}{\sqrt{21}}, \quad z = \pm\frac{32}{\sqrt{21}}$$

všechny kombinace - maxima.

Otázka na p. Běhounkovou nebo někoho:

→ Ve vzorovém řešení pak ještě byly rozebírány případy $x=0, y=0, z=\pm\frac{32}{\sqrt{21}}$, atd., což je asi dáno tím, že $\operatorname{sign}|x|$ a tak může být nula, je to ale nutné? Teď to sám nedokážu říct.

→ Je to nutné. Takto to nestačí!

DODĚLAT! Není to těžké!

2019 - Početní 5

Regulární zobrazení + Jakobián

(3) 6b Přechodem k polárním souřadnicím $x = r\cos\varphi$, $y = r\sin\varphi$

vyjádřete $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ tj přejděte k fci $\tilde{u}(r,\varphi) := u(x(r,\varphi), y(r,\varphi))$

a výraz přepište pomocí fce $\tilde{u}$ a jejích derivací.

Nezapomeňte ověřit, na jaké množině jsou splněny předpoklady globální věty o inverzi.

$$\Rightarrow \frac{\partial \tilde{u}}{\partial r} = \frac{\partial u}{\partial x}\cos\varphi + \frac{\partial u}{\partial y}\sin\varphi$$

$$\frac{\partial \tilde{u}}{\partial \varphi} = \frac{\partial u}{\partial x}(-r\sin\varphi) + \frac{\partial u}{\partial y}r\cos\varphi$$

$$\frac{\partial \tilde{u}}{\partial r} - \frac{\partial u}{\partial y}\sin\varphi = \frac{\partial u}{\partial x}\cos\varphi$$

$$\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial r}\frac{1}{\cos\varphi} - \frac{\partial u}{\partial y}\frac{\sin\varphi}{\cos\varphi}$$

$$\frac{\partial \tilde{u}}{\partial r} - \frac{\partial u}{\partial x}\cos\varphi = \frac{\partial u}{\partial y}\sin\varphi$$

$$\frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial r}\frac{1}{\sin\varphi} - \frac{\partial u}{\partial x}\frac{\cos\varphi}{\sin\varphi}$$

$$\frac{\partial u}{\partial x}(-r\sin\varphi) = \frac{\partial \tilde{u}}{\partial \varphi} - \frac{\partial u}{\partial y}r\cos\varphi$$

$$\frac{\partial u}{\partial x} = -\frac{\partial \tilde{u}}{\partial \varphi}\frac{1}{r\sin\varphi} + \frac{\partial u}{\partial y}\frac{\cos\varphi}{\sin\varphi}$$

$$\frac{\partial u}{\partial y}r\cos\varphi = \frac{\partial \tilde{u}}{\partial \varphi} + \frac{\partial u}{\partial x}r\sin\varphi$$

$$\frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial \varphi}\frac{1}{r\cos\varphi} + \frac{\partial u}{\partial x}\frac{\sin\varphi}{\cos\varphi}$$

dosadím

dosadím

$$\frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial \varphi}\frac{1}{r\cos\varphi} + \frac{\sin\varphi}{\cos\varphi}\left(\frac{\partial \tilde{u}}{\partial r}\frac{1}{\cos\varphi} - \frac{\partial u}{\partial y}\frac{\sin\varphi}{\cos\varphi}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial \tilde{u}}{\partial \varphi}\frac{1}{r\sin\varphi} + \frac{\cos\varphi}{\sin\varphi}\left(\frac{\partial \tilde{u}}{\partial r}\frac{1}{\sin\varphi} - \frac{\partial u}{\partial x}\frac{\cos\varphi}{\sin\varphi}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\cos^2\varphi}{\sin^2\varphi} = -\frac{\partial \tilde{u}}{\partial \varphi}\frac{1}{r\sin} + \frac{\partial \tilde{u}}{\partial r}\frac{\cos\varphi}{\sin^2\varphi} \quad / \cdot \sin^2\varphi$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial r}\cos\varphi - \frac{\partial \tilde{u}}{\partial \varphi}\frac{\sin\varphi}{r}}$$

$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\frac{\sin^2\varphi}{\cos^2\varphi} = \frac{\partial \tilde{u}}{\partial \varphi}\frac{1}{r\cos\varphi} + \frac{\partial \tilde{u}}{\partial r}\frac{\sin\varphi}{\cos^2\varphi} \quad / \cdot \cos^2\varphi$$

$$\boxed{\frac{\partial u}{\partial y}\underbrace{(\cos^2\varphi + \sin^2\varphi)}_{1} = \frac{\partial \tilde{u}}{\partial \varphi}\frac{\cos\varphi}{r} + \frac{\partial \tilde{u}}{\partial r}\sin\varphi}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = r\cos\varphi\left(\frac{\partial \tilde{u}}{\partial r}\cos\varphi - \frac{\partial \tilde{u}}{\partial \varphi}\frac{\sin\varphi}{r}\right) + r\sin\varphi\left(\frac{\partial \tilde{u}}{\partial \varphi}\frac{\cos\varphi}{r} + \frac{\partial \tilde{u}}{\partial r}\sin\varphi\right)$$

$$= r\frac{\partial \tilde{u}}{\partial r}$$

(tu finální úpravu vidím ve vzorovém řešení taky, ale i mo to řeší)

další str. ověření

Ověření předpokladů o reg. zobrazení

str. 202 skripta

(Jacobiho matice)

Jakobián

$$M = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\varphi) & \frac{\partial}{\partial \varphi}(r\cos\varphi) \\ \frac{\partial}{\partial r}(r\sin\varphi) & \frac{\partial}{\partial \varphi}(r\sin\varphi) \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

determinant? $\det M = r\cos^2\varphi + r\sin^2\varphi$

$= r$

$r \neq 0$ mimo počátek

$(x,y) \neq (0,0)$

Zároveň je pro $r>0$, $\varphi \in (0, 2\pi)$ na <u>otevřené</u> množině prostá.

2015 – Početní 2

impl. fce

$\left(\frac{\partial z}{\partial x} z\right)'$

(4) 8b

Ověřte, že implicitně zadaná funkce $z = z(x,y)$ vyčerní pomocí

$z^3 + (x-1)e^z + \sin\left(\frac{\pi x}{2}\right) - zy\ln y = 0$

na nějakém okolí bodu $(1,1,-1)$ je dobře definovaná.

Spočtěte $\frac{\partial^2 z}{\partial x^2}(1,1)$ a $\frac{\partial^2 z}{\partial y^2}(1,1)$

① $F_1\big|_{(1,1,-1)} = -1 + 0 \cdot e^{\cdot} + \sin\left(\frac{\pi}{2}\right)^{\nearrow 1} - 1 \cdot \ln 1^{\nearrow 0} = 0$ ✓

② $\frac{\partial F}{\partial z} = 3z^2 + (x-1)e^z - y\ln y$

$\big|_{(1,1,-1)} = 3 + 0 - 1\ln 1 = 3 \neq 0$ ✓

③ Fce je hladké.

$\frac{\partial F_{(x,y)}}{\partial x} = 3z^2 z' + (x-1)e^z z' \; {+e^z} + \cos\left(\frac{\pi x}{2}\right) \cdot \frac{\pi}{2} - z'y\ln y \cancel{+ z\ln y} \cancel{+ z\frac{y}{y}}$

pro $(0,0) = 3z' \; {0+e^{-1}} \cancel{-1} = 0 \rightarrow \frac{\partial z}{\partial x} = \cancel{\frac{1}{3}} \varnothing \frac{1}{3e}$

$\frac{\partial^2 F}{\partial x^2} = 6zz' + 3z^2 z'' + \underbrace{(x-1)e^z \ldots}_{0} \ldots - \sin\left(\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)^2 - \ldots z''y\ln y \cancel{\frac{z}{y}} \cancel{z'}$ $+ z'e^z + e^z z' + \ldots 0$

$= -\cancel{\frac{6}{3}} + 3z'' - \left(\frac{\pi}{2}\right)^2 \cancel{+1 - 1} + \cancel{1} = \cancel{-1} + 3z'' - \frac{\pi^2}{4} \longrightarrow z'' = \frac{1 + \frac{\pi^2}{4}}{}$

$= \frac{-2}{e} + 3z'' - \frac{\pi^2}{4} \quad + \frac{1}{3e e \cdot 2}$

$= \frac{\pi^2}{4 \cdot 3}$ ☹ OOO

$\frac{\partial F}{\partial y} = 3z^2 z' + (x-1)e^z z' - z'y\ln y - z\ln y - z\frac{y}{y}$

$\big|_{(0,0)} = 3z' - z = 0 \rightarrow z' = -\frac{1}{3}$

$\frac{\partial^2 F}{\partial y^2} = 6zz' + 3z^2 z'' - z'\frac{y}{y} - z'\frac{1}{y} - z' =$

$\big|_{(0,0)} \; 2 + 3z'' + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} \longrightarrow 3z'' + \frac{9}{3} = 0 \rightarrow z'' = -\cancel{\frac{7}{9}} \; 1$

2015 – Početní 1 impl. fce

③ 8b Nechť $x^2 e^{u+v} + 2u^2v^2 = 1$

$y^2 e^{u-v} - \frac{u}{(1+v)^2} = 4x$

ukažte že na okolí bodu $x = y = 2$ $u = 0 = v$ určuje výraz ∃ fcí $u(x,y), v(x,y)$.

Určete $du(1,2)$ a $dv(1,2)$

① $F_1|_{(1,2,0,0)}$ $1 \cdot e^0 + 0 = 1$ ✓

$F_2|_{(1,2,0,0)}$ $4 \cdot e^0 - 0 = 4 \cdot 1$ ✓

② $\cancel{\frac{\partial F_1}{\partial x} = 2xe^{u+v} \quad \frac{\partial F_1}{\partial y} = 0 \quad \frac{\partial F_2}{\partial x} = -4 \quad \frac{\partial F_2}{\partial y} = 2ye^{u-v}}$

$\cancel{\rightarrow \begin{pmatrix} 2xe^{u+v} & 0 \\ -4 & 2ye^{u-v} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -4 & 4 \end{pmatrix} = 8 - 0 \neq 0}$

POZOR! $\frac{\partial F_1}{\partial u}, \frac{\partial F_2}{\partial u}$ $\frac{\partial F_1}{\partial v}$ $\frac{\partial F_2}{\partial v}$ NE x a y!

③ Fce je hladká 1 řádu a kopř. C^1.

$\Rightarrow \frac{\partial f_1}{\partial x} = 2xe^{u+v} + x^2e^{u+v}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) + 4uv^2\frac{\partial u}{\partial x} + 4u^2v\frac{\partial v}{\partial x} = 0$

$\frac{\partial f_2}{\partial x} = y^2e^{u-v}\cdot\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) - \frac{\partial u}{\partial x}(1+v)^{-2} - u\cdot(-2)(1+v)^{-3}\cdot\frac{\partial v}{\partial x} = \cancel{0}\ 4$

$= 2 + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \rightarrow \frac{\partial v}{\partial x} = -2 - \frac{\partial u}{\partial x}$

$= 4\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right) - \frac{\partial u}{\partial x} = \cancel{0}\ 4$ $\quad 4\left(\frac{\partial u}{\partial x} + 2 + \frac{\partial u}{\partial x}\right) - \frac{\partial u}{\partial x} = 7\frac{\partial u}{\partial x} \cancel{+2}\ {+4} = 0$

$\rightarrow \frac{\partial u}{\partial x} = -\frac{4}{7}$

$\frac{\partial v}{\partial x} = -2 + \frac{4}{7} = \frac{-14+4}{7}$

$\frac{\partial f_1}{\partial y} = x^2e^{u+v}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) + \left(\text{po dosazení bude } 0\right) = 0$

$\frac{\partial f_2}{\partial y} = 2ye^{u-v} + y^2e^{u-v}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y}(1+v)^{-2} + (\text{bude } 0) = 0$

$= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 \rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial y}$

$= 4 + 4\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}\right) - \frac{\partial u}{\partial y} = 0$ $\quad 4 + 4\cdot 2\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} = 0$

$7\frac{\partial u}{\partial y} = -4$

$\frac{\partial u}{\partial y} = -\frac{4}{7}$

$\frac{\partial v}{\partial y} = \frac{4}{7}$

$\nabla u(1,2) = \left(-\frac{4}{7}, -\frac{4}{7}\right)$ $\quad \nabla v(1,2) = \left(-\frac{10}{7}, \frac{4}{7}\right)$

$du(1,2)(h_1,h_2) = -\frac{4}{7}h_1 - \frac{4}{7}h_2$

$dv(1,2)(h_1,h_2) = -\frac{10}{7}h_1 + \frac{4}{7}h_2$

vazba

2015 - Početní 1

(4) 8b Největší a nejmenší hodnota na elipsoidu

$$f(x,y,z) = x^6 y^6 z^6$$

vazba $x^2 + 2y^2 + 3z^2 \leq 1$

vnitřek

$$\frac{\partial f}{\partial x} = 6x^5y^6z^6 \quad \frac{\partial f}{\partial y} = 6x^6y^5z^6 \quad \frac{\partial f}{\partial z} = 6x^6y^6z^5$$

$\rightarrow$ možné minimum v $(0,0,0)$ není ale jediné. kdykoliv $x=0 \wedge y=0 \wedge z=0$ $f(x,y,z)=0$

$\rightarrow$ skončíme, vzhledem k tomu, že $(y^2)^3$ a analogicky x a y jsou vždy kladné, na Df neexistuje nižší hodnota než $f(0,0,0)=0$.

vazba

$$f(x,y,z,\lambda) = x^6y^6z^6 - \lambda(x^2+2y^2+3z^2-1)$$

$$\frac{\partial f}{\partial x} = 6x^5y^6z^6 - 2\lambda x$$

$$\frac{\partial f}{\partial y} = 6x^6y^5z^6 - 4\lambda y$$

$$\frac{\partial f}{\partial z} = 6x^6y^6z^5 - 6\lambda z$$

$6x^5y^6z^6 - 2\lambda x = 0$

$6x^6y^5z^6 - 4\lambda y = 0$

$6x^6y^6z^5 - 6\lambda z = 0$

$x^2 + 2y^2 + 3z^2 = 1$

$2\lambda x^2 = 4\lambda y^2 = 6\lambda z^2$

$x^2 = 2y^2 = 3z^2$

$3x^2 = 1$

$x = \pm \frac{1}{\sqrt{3}}$

$x^2 = 2y^2 = 3z^2 = \frac{1}{3}$

$2y^2 = \frac{1}{3} \rightarrow y = \pm\frac{1}{\sqrt{6}}$

$3z^2 = \frac{1}{3} \rightarrow z = \pm\frac{1}{3}$

glob. max. v $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{6}}, \pm\frac{1}{3}\right)$

pro všechny kombinace znamének.

vazba

2015 - Početní 1

3) 7b Největší a nejmenší hodnota funkce

$$f(x,y,z) = x^2 + 2y^2 - 3z^2$$

na elipsoidu $3x^2 + 2y^2 + z^2 \leq 1$

UVNITŘ $\frac{\partial f}{\partial x} = 2x$ $\frac{\partial f}{\partial y} = 2y$ $\frac{\partial f}{\partial z} = -3z$ → možné minima v $(0,0,0)$ - není to ale nejmenší hodnota. Např. $(0,0,1)$ - menší hodnota. $(0,1,0)$ - větší hodnota

$\Longrightarrow$ Max a min se nabývá na okraji vazby $3x^2+2y^2+z^2=1$

VAZBA

$$f(x,y,z,\lambda) = x^2 + 2y^2 - 3z^2 - \lambda(3x^2 + 2y^2 + z^2 - 1)$$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x - \lambda 6x \\ \frac{\partial f}{\partial y} &= 4y - \lambda 4y \\ \frac{\partial f}{\partial z} &= -6z - \lambda 2z \end{aligned}\right\} \quad \left.\begin{aligned} 2x - \lambda 6x &= 0 \\ 4y - \lambda 4y &= 0 \\ -6z - \lambda 2z &= 0 \\ 3x^2+2y^2+z^2-1 &= 0 \end{aligned}\right\} \quad \begin{aligned} 2x(1-3\lambda) &= 0 \\ 4y(1-\lambda) &= 0 \\ -2z(3+\lambda) &= 0 \end{aligned}$$

1) $\lambda = \frac{1}{3}$ $y = z = 0$ → $3x^2 = 1$ $x = \frac{1}{\sqrt{3}}$ → $f = \frac{1}{3}$

2) $\lambda = 1$ $x = z = 0$ → $2y^2 = 1$ $y = \frac{1}{\sqrt{2}}$ → $f = 1$ MAX

3) $\lambda = -3$ $x = y = 0$ → $z^2 = 1$ $z = 1$ → $f = -3$ MIN

tedy: maximum je v $(0, \pm\frac{1}{\sqrt{2}}, 0)$

minimum je v $(0, 0, \pm 1)$

2015 - Početní 2

extrémy

③ 8b lok extrémy fce + klasifikace

$$f(x,y) = (5x+7y-25)e^{-(x^2+xy+y^2)} \quad \text{def na } \mathbb{R}^2$$

$$\frac{\partial f}{\partial x} = 5e^{-(\;)} + (5x+7y-25)\cdot(-2x-y)e^{-(\;)}$$

$$\frac{\partial f}{\partial y} = 7e^{-(\;)} + (5x+7y-25)\cdot(-2y-x)e^{-(\;)}$$

$$5+(5x+7y-25)(-2x-y) = 7+(5x+7y-25)(-2y-x)$$

$$-10x^2-14y+50x-5xy-7y^2+25y = \dots -5$$

$$(5x+7y-25)(-2x-y) = -5 \quad /7$$

$$(5x+7y-25)(-2y-x) = -7 \quad /5$$

$$\rightarrow \quad -14x-7y = -10y-5x$$

$$3y = 9x$$

$$\underline{y = 3x}$$

$$5+(5x+21x-25)(-2x-3x) = 0$$

$$5-(26x-25)(5x) = 0$$

$$5(1-(26x-25)x) = 0$$

$$26x^2-25x+1 = 0$$

$$(26x+1)(x-1) = 0$$

$$x_1 = 1 \qquad x_2 = -\frac{1}{26}$$

$$y_1 = 3, \qquad y_2 = -\frac{3}{26}$$

$$\frac{\partial^2 f}{\partial x^2} = \left(e^{-(\;)}\cdot(5-10x^2\underbrace{-5xy-14xy}_{-19xy}-7y^2+50x+25y)\right)'$$

$$= e^{-(\;)}\cdot\left[(-20x-19y+50)+(5-10x^2-19xy-7y^2+50x+25y)(-2x-y)\right]$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y} = e^{-(\;)}\left[(-19x-14y+25)+(\;\;\ldots\;\;)(-2y-x)\right]$$

$$\frac{\partial^2 f}{\partial y^2} = \left(e^{-(\;)}\cdot\left[7-10xy-5x^2-14y^2-7xy+50y+25x\right]\right)'$$

$$= e^{-(\;)}\cdot\left[-17x-28y+50+(7-10xy\ldots)(-2y+x)\right]$$

nebot fce je třídy na C^2

pro $x_1, y_1 = 3$

$$\frac{\partial^2 f}{\partial x^2} = e^{-(1+3+9)}\left(\underbrace{-20-19\cdot3+50}_{-27}+(\underbrace{5-10-19\cdot3-7\cdot9+50+75}_{-62-63+125})(-5)\right) = \boxed{-27}$$

$$\frac{\partial^2 f}{\partial x\partial y} = (-19-42+25) = \boxed{-36}$$

$$\frac{\partial^2 f}{\partial y^2} = (-17\underbrace{-28\cdot3}_{-84}+50) = \boxed{-51}$$

$$\begin{pmatrix} -27 & -36 \\ -36 & -51 \end{pmatrix}$$

$a_1 = -51$

$a_2 = +$

$27\cdot51 > 36^2$

neg def

lok MAX

72

Bernoulliova rovnice

S03

$y' - xy = -y^3 e^{-x^2}$

$y' y^{-3} - x \frac{1}{y^2} = -e^{-x^2}$

$z(x) = y^{-2}$ --- $\frac{\partial z}{\partial x} = -2 y^{-3} \cdot y'$

$y' = \frac{1}{-2} y^3 z'$

$\dashrightarrow \quad z'\left(-\frac{1}{2}\right) - xz = -e^{-x^2}$

$z_h = ?$

$-z' \frac{1}{2} = xz$

$z' = -2xz$

$\frac{\partial z}{z} = -2x\, \partial x$

$\ln|z| = -x^2 + c$

$z = Ce^{-x^2}$

$z_p = ?$ $\quad z_p = C(x) e^{-x^2}$

$-\frac{1}{2} \cdot C(x)' e^{-x^2} + \cancel{\frac{1}{2} 2x e^{-x^2} \cdot C(x) - xz} = -e^{-x^2}$

$C'(x) e^{-x^2} = 2e^{-x^2}$

$C'(x) = +2$

$C(x) = 2x$

$z = Ce^{-x^2} + 2x$

$y^{-2} = Ce^{-x^2} + 2x$

$y^2 = \frac{1}{ce^{-x^2} + 2x}$

$y = \sqrt{\frac{1}{\dots}}$

2015 - Početní část 2

ŘADY

② 6b Dokažte, zda konverguje řada

$$\sum_{n=2}^{\infty} \underbrace{\frac{1}{\ln(\sqrt{n+1})}}_{\sum_{n=2}^{\infty} a_k} \underbrace{\cos\left(\frac{n^3\pi}{n^2+1}\right)}_{\sum_{n=2}^{\infty} b_k}$$

← Nápověda (viz disk)

→ dospěj k Leibnitzovu kritériu

$\sum b_k$ → omezené částečné součty

$$\cancel{\frac{\partial f}{\partial x} \frac{1}{\ln(\sqrt{x+1})} = \left(\frac{1}{\sqrt{x}+1} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}\right)^{-1} = \left(\frac{1}{\sqrt{x}+1} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x}}\right)^{-1} \ldots \text{pro } n \to \infty \text{ jde}}$$

$$= 2(\sqrt{x}+1)\sqrt{x}$$

$$\frac{\partial f}{\partial x} \left(\ln(\sqrt{x+1})\right)^{-1} = -1 \left(\ln(\sqrt{x+1})\right)^{-2} \cdot \frac{1}{\sqrt{x}+1} \cdot \frac{1}{2} \cdot x^{\frac{1}{2}} \to \text{monotónní}$$

$$\lim_{n \to \infty} \frac{1}{\ln(\sqrt{n+1})} = 0$$ protože $\ln(\sqrt{n+1}) \xrightarrow{n\to\infty} = \infty$

⇒ dle Dirichleta K

dle srovnávacího kritéria

$$\frac{1}{\ln(\sqrt{n+1})} \left|\cos\left(\frac{n^3\pi}{n^2+1}\right)\right| \geq \frac{\varepsilon}{\ln(\sqrt{n+1})} \geq \varepsilon \cdot \frac{1}{n} \ldots D \qquad \varepsilon \in (0,1)$$

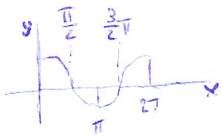
⇒ nekonverguje absolutně

2015 - Početní část 11.6

② 6b V závislosti na $x \in [0, 2\pi]$ vyšetřete konvergenci řady

$$\sum_{n=1}^{\infty} \frac{\sqrt{1+n^2}\cos(nx)}{n^2\ln(n^2+1)} \longrightarrow \sum_{n=1}^{\infty} \frac{\sqrt{1+n^2}}{n^2\ln(n^2+1)} = \sum a_k$$

$$\sum_{n=1}^{\infty} \cos(nx) = \sum b_k$$



pro $x = 0, 2\pi \ldots \cos(nx) = 1$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2\ln(n^2+1)} \ldots K? \ldots = \sum_{n=1}^{\infty} \frac{\sqrt{\overbrace{n(1+\tfrac{1}{n})}^{\text{pro } n\to\infty \to 1}}}{n^2\ln(n^2+1)} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}\ln(n^2+1)}$$ dále dle srovnávacího kritéria

$$\exists n_0 \; \forall n > n_0 \quad \frac{1}{n^{\frac{3}{2}}\ln(n^2+1)} \leq \frac{1}{n^{\frac{3}{2}}}$$

a $\frac{1}{n^{3/2}}$ dle int. krit. K

⟹ pro $x = 0, 2\pi$ řada K dokonce absolutně

pro $x \in (0, 2\pi)$

$\sum a_k$ je monotónní, omezená, klesající, $\lim_{n\to\infty} a_k = 0$

$\sum b_k$ omezené část. součty

} dle Dirichleta K

2014 - POČETNÍ ČÁST 4

ŘADY staré

① $\bullet \sum_{n=10}^{\infty} \sin\left(\frac{1}{n^3} + \frac{\cos(n)}{n^2}\right) = \sum_{n=10}^{\infty} \sin\left(\frac{1+n\cos(n)}{n^3}\right)$

pro $n \to \infty$ $n^3 >> 1 + n\cos(n)$ dle srovnávacího kritéria

tedy $\lim_{n\to\infty} \frac{1+n\cos(n)}{n^3} = 0$

z identity $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ můžeme použít

$\cos(n) \in [-1, 1]$

$$\sum_{n=10}^{\infty} \sin\left(\frac{1+n\cos(n)}{n^3}\right) = \sum_{n=10}^{\infty} \frac{1+n\cos(n)}{n^3} = \sum_{n=10}^{\infty} \underbrace{\frac{1}{n^3}}_{\text{K dle int krit absolutně}} + \sum_{n=10}^{\infty} \underbrace{\frac{1}{n^2}}_{\text{monotónní jde k nule}} \overbrace{\cos(n)}^{\text{Omezené částečné součty}}$$

K dle int. krit

K dle Dirichleta

navíc $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2} < \sum_{n=1}^{\infty} \frac{\varepsilon}{n^2}$ $\varepsilon > 1$

$\to$ abs. K

součet dvou K řad je K řada z aritmetiky řad

$\Rightarrow$ řada K absolutně

$\bullet \sum_{n=2}^{\infty} \underbrace{\sin(n)}_{\text{Omezené částečné součty}} \underbrace{\frac{\lg(10n)}{n}}_{K ??}$

monotónní klesá k nule

integrální krit. předpoklady; monotónní, $\to 0$, splněny

$$\int_2^{\infty} 10 \frac{\ln(10n)}{10n}\, dn \quad \left| \begin{array}{l} \ln(10n) = t \\ \frac{1}{10n}\cdot 10 = dt \\ dn \\ \hookrightarrow dt = 10 \frac{1}{n} \end{array} \right|$$

$$= \int_{\ln 2}^{\infty} \frac{1}{10} t\, dt = \left[\frac{1}{20} t^2\right]_{\ln 2}^{\infty} = \infty$$

$\to$ Diverguje

$\lim_{n\to\infty} \frac{1}{n} \ln(10n) = 0$ protože pro $n \to \infty$ $n >> \ln(n)$

$\Rightarrow$ Řada neabsolutně K díky Dirichletovi

vyzkoušíme absolutní K

$$\sum_{n=2}^{\infty} \left| \underbrace{\sin(n)}_{\in (0,1) \text{ kvůli té absolutní}} \underbrace{\frac{\ln(10n)}{n}}_{\text{vždy kladné}} \right| \leq \sum_{n=2}^{\infty} \varepsilon \frac{\ln(10n)}{n}$$

$\varepsilon > 1$

a to jsme výše ukázali že diverguje.

$\Rightarrow$ Řada nekonverguje absolutně

2014 – POČETNÍ ČÁST 2 ŘADY

① $$\sum_{n=1}^{\infty} \frac{\lg(n) - \lg(n-1)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\ln\left(\frac{n}{n-1}\right)}{\sqrt{n}} \to \lim_{n\to\infty} a_k = 0 \text{ .. splněna nutná podm. konv.}$$

$$= \sum_{n=1}^{\infty} \frac{\ln\left(\frac{n-1}{n-1} + \frac{1}{n-1}\right)}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\ln\left(1 + \frac{1}{n-1}\right)}{\sqrt{n}}$$

víme, že $\lim_{n\to 0} \frac{\ln(n+1)}{n} = 1$

tedy $\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n-1}\right)}{\frac{1}{n-1}} = 1$

$$= \sum_{n=1}^{\infty} \frac{\frac{1}{n-1}}{\sqrt{n}}$$ pro dostatečně velké n

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)\sqrt{n}} \underset{\text{díky srovnávacímu kritériu}}{\sim} \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \to$$ dle integrálního kritéria konverguje.

② $$\sum_{n=1}^{\infty} \sin\left(\frac{\pi n^2 \lg(n)}{n\lg(n) - 1}\right)$$ ..... je splněna nutná podm. k? Jak se chová ta věc uvnitř sin(...)

$$\lim_{n\to\infty} \frac{\pi n^2 \ln(n)}{n\ln(n) - 1} = \lim_{n\to\infty} \pi \frac{\overbrace{n \ln(n)}^{\to\infty\ \to\infty}}{\underbrace{\ln(n)}_{\to\infty} - \underbrace{\frac{1}{n}}_{\to 0}} \sim \lim_{n\to\infty} \frac{n\ln(n)}{\ln(n) - \underbrace{\varepsilon}_{\text{nezajímavý}}} \cdot \pi = \lim_{n\to\infty} n \overbrace{\frac{\ln(n)}{\ln(n) - \varepsilon}}^{1 \text{ pro } n\to\infty} \cdot \pi = \infty$$

Vnitřek toho sin tedy roste

přes všechny meze.

→ Řada osciluje.

2015 – Početní část 5.6.

Řady

$\boxed{\text{mocninné}}$

② [8b] Zjistěte, na jaké podmnožině $\mathbb{C}$ konverguje mocninná řada

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} z^n$$

Pak uvažujte průnik řady s reálnou osou. Na ní řadu sečtěte + odůvodněte vše.

Poloměr konvergence?

$$R = \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{(n+1)(n+2)}{n(n+1)}\right| = \frac{n+2}{n} = \frac{\frac{1}{n}(1+\frac{2}{n})}{\frac{1}{n}(1+\frac{1}{n})} \overset{\text{pro } n\to\infty}{=} \frac{n}{n} = 1$$

$R = 1$ řada absolutně K pro $|z| < 1$

D pro $|z| > 1$

$|z| = 1$ ... ? Vyšetříme zvlášť. $|z| = e^{i\varphi}|R| = e^{i\varphi}R$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} e^{i\varphi n} = \sum_{n=1}^{\infty} \underbrace{\frac{1}{n(n+1)}}(\cos n\varphi + i\sin n\varphi)$$

díky srovnávacímu kritériu můžeme uvažovat řadu $\frac{1}{n^2}$. U té díky integrálnímu krit. víme, že K.

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{K} \underbrace{(\cos n\varphi + i\sin n\varphi)}_{\text{omezená, omezené částečné součty}}$$

ani $\varphi = 0$ není problém, jelikož $\sum_{n=1}^{\infty} \frac{1}{n^2} \cdot 1^n$ K

pro $\varphi \in [0, 2\pi]$ K, dokonce absolutně.

dle Dirichleta

- Součet: součet mocninné řady $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$

2019 - Početní část 2

ODR

(1) Nalezněte maximální řešení rovnice $y''' - y'' + y' - y = e^x + e^{-x}$

poč. podm. $y(0) = -\frac{1}{4}$, $y'(0) = \frac{11}{4}$, $y''(0) = \frac{11}{4}$

char. polynom: $\lambda^3 - \lambda^2 + \lambda - 1 = 0 \quad / : \lambda + 1$

$$\lambda^3 - \lambda^2 + \lambda - 1 : \lambda + 1 = \lambda^2 - 2\lambda$$
$$-\lambda^3 - \lambda^2$$
$$-2\lambda^2 + \lambda$$
$$+2\lambda + 2\lambda$$
$$3\lambda - 1 \ \ldots \text{NE}$$

$$\lambda^3 - \lambda^2 + \lambda - 1 : \lambda - 1 = \lambda^2 + 1$$
$$-\lambda^3 + \lambda^2$$
$$0 \quad \lambda - 1$$
$$-\lambda + 1$$
$$0$$

$$= (\lambda - 1)(\lambda^2 + 1) \longrightarrow FS = (e^x, e^{-ix}, e^{ix})$$

$$y_h(x) = C_1 e^x, C_2 \sin x, C_3 \cos x = C_1 e^x + C_2 \sin x + C_3 \cos x$$

$$y_p(x) = Axe^x, Be^{-x}$$

.... Axe^x .. $A = ?$

$$-Axe^x + (Axe^x)' - (Axe^x)'' + (Axe^x)''' = 0$$

$$-\cancel{Axe^x} + (\cancel{Ae^x} + Ax^2e^x) - (Axe^x + A2xe^x + Ax^3e^x) + (Ae^x + Ax^2e^x + 2Ae^x + 2Ax^2e^x + A3x^2e^x + Ax^4e^x)$$

$$-4 \qquad +4 \qquad 7$$

$$= -Axe^x - Axe^x - A2xe^x + Ae^x + 3Ae^x + Ax^2e^x + Ax^2e^x + A5x^2e^x + Ax^3e^x + Ax^4e^x$$

$$= e^x A(-4x + 4 + 7x^2 + x^3 + x^4) = Ae^x(x^4 + x^3 + 7x^2 - 4x + 4)$$

ZNOVU

$$(Axe^x)' = e^x A(1 + x^2)$$
$$(Axe^x)'' = e^x A(2x + x + x^3) = e^x A(3x + x^3)$$

NAUČ SE DERIVOVAT!

$$(Axe^x)''' = e^x A(3 + 3x^2 + 3x^2$$
$$(Axe^x)' = Ae^x + xAe^x = Ae^x(1 + x)$$
$$(Axe^x)'' = Ae^x(1 + 1 + x) = Ae^x(2 + x)$$
$$(Axe^x)''' = Ae^x(1 + 2 + x) = Ae^x(3 + x)$$
$$\rightarrow Ae^x(-\cancel{x} + 1 + \cancel{x} - 2 - \cancel{x} + 3 + \cancel{x}) = Ae^x 2 = 1e^x \rightarrow A = \frac{1}{2}$$

$$\rightarrow Be^{-x}(-1 - 1 - 1 - 1) = -4Be^{-x} = e^{-x} \rightarrow B = -\frac{1}{4}$$

$\rightarrow$ řešení $y_p + y_h = C_1 e^x + C_2 \sin x + C_3 \cos x + \frac{1}{2}e^x - \frac{1}{4}e^{-x}$

$$\left.\begin{aligned} C_1 + C_2 - \tfrac{1}{4} &= -\tfrac{1}{4} \\ C_1 + C_3 + \tfrac{1}{2} + \tfrac{1}{4} &= \tfrac{11}{4} \\ C_1 - C_2 + 1 - \tfrac{1}{4} &= \tfrac{11}{4} \end{aligned}\right\}$$

$$2C_1 + \frac{1}{2} = \frac{5}{2}$$

$C_1 = 1 \quad C_2 = -1 \quad C_3 = 1$

$$y(x) = e^x - \cos x + \sin x + \frac{1}{2}xe^x - \frac{1}{4}e^{-x}$$

Řady rozjezd

(9) $\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{\ln \ln n}}$

$(\ln n)^{\ln \ln n} = e^{(\ln \ln n)^2}$

$\ln x$ roste pomaleji než x^α $\alpha > 0$

a tedy: $\exists x_0\ \forall x > x_0\ \ln x \leq x^\alpha$ např. $\alpha = \frac{1}{2}$

$\rightarrow \ln x \leq x^{1/2}$

$e^{(\ln \ln n)^2} \leq e^{(\sqrt{\ln n})^2} = e^{\ln n} = n$

a tedy $\frac{1}{e^{(\ln \ln n)^2}} \geq \frac{1}{n}$

$\sum_{n=3}^{\infty} \frac{1}{n}$ diverguje $\frac{1}{(\ln n)^{(\ln \ln n)}} \geq \frac{1}{n}$ pro $\forall n > n_0$ a tedy

$\sum_{n=3}^{\infty} \frac{1}{(\ln n)^{(\ln \ln n)}}$ D

(10) $\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$ → nutná podm. konv.? $\lim_{n\to\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{e}} = 1$

NENÍ SPLNĚNA

Řada D

① ... ⑥ .. součty řad, elementární úvahy. Nebude v testu. 505

Řady rozjezd

⑦ $\sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1}-\sqrt{n-1}\right)$ $\nearrow$ $\lim_{k\to\infty} a_k = 0$ nutná podm. splněna

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1}-\sqrt{n-1}\right) \Big/ \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \frac{n+1-n+1}{\sqrt{n+1}+\sqrt{n-1}} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{2}{\sqrt{n+1}+\sqrt{n-1}}$$

$\longrightarrow$ limitní srovnávací kritérium s $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = b_k$

$$\lim_{n\to\infty} \frac{a_k}{b_k} = \lim_{n\to\infty} 2\frac{1}{\underbrace{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}_{n\to\infty\ \sim 2\sqrt{n}}} \cdot n^{3/2} = \underline{1}$$

potřebujeme aby vyšlo konečné číslo $(0,\infty)$

$\longrightarrow$ můžeme tedy dále pracovat s $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. $\alpha = 3/2 > 1$

$\rightarrow$ řada K

Ještě ověříme integrálním krit.

spojitá nerostoucí kladná

$$\int_1^{\infty} n^{(-3/2)}\, dx = \int_1^{\infty} x^{-3/2}\, dx = \left[x^{-1/2}\cdot(-2)\right]_1^{\infty} = \left(\frac{1}{\sqrt{1}}\cdot(-2) - \frac{1}{\sqrt{\infty}}\cdot(-2)\right)$$

$= -2 \rightarrow$ konečné

$\rightarrow$ řada K

⑧ $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$ $\nearrow$ $\lim_{k\to\infty} a_k = 0$ nutná podm. splněna

jaké c zvolím?

víme, že $(\ln n)^{\ln n} = e^{\ln n(\ln \ln n)} > e^{c(\ln n)}$

pro dostatečně velké n ?

Tedy:

$$\frac{1}{(\ln n)^{\ln n}} < \frac{1}{e^{\ln n\, c}} = \frac{1}{n^c}$$

$\ln\ln n > c$

$\ln n > e^c$

$n > e^{e^c}$

pro $c > 1$ řada $\sum_{n=1}^{\infty} \frac{1}{n^c}$ K.

zvolme např. $c=2$. z lim. srovnávacího kritéria

$$\lim_{n\to\infty} \frac{n^2}{(\ln n)^{\ln n}} = 0 \longrightarrow \text{řada K}$$

2015 - Početní 1 ODR

① 2b

Naleznete obecné řešení ODR $y''' - 3y'' - 4y' = x^3 + xe^x$

$\lambda^3 - 3\lambda^2 - 4\lambda = 0$

$\lambda(\lambda^2 - 3\lambda - 4) = \lambda(\lambda - 4)(\lambda + 1) = 0 \quad \lambda_1 = 0 \quad \lambda_2 = 4 \quad \lambda_3 = -1$

$y_h(x) = C_1 e^{0x} + C_2 e^{4x} + C_3 e^{-x} = C_1 + C_2 e^{4x} + C_2 e^{-x}$

$y_p(x) = ?$

• Nejprve x^3: polynom je $Ax^3 + Bx^2 + Cx + D$

→ jelikož $\lambda_1 = 0$, máme $Ax^4 + Bx^3 + Cx^2 + Dx$

$y = Ax^4 + Bx^3 + Cx^2 + Dx$

$y' = 4Ax^3 + 3Bx^2 + 2Cx + D$

$y'' = 12Ax^2 + 6Bx + 2C$

$y''' = 24Ax + 6B$

$(24Ax + 6B) - 3(12Ax^2 + 6Bx + 2C) - 4(4Ax^3 + 3Bx^2 + 2Cx + D) = x^3$

$\rightarrow 24Ax - 18Bx + 2Cx = 0$

$6B - 6C - 4D = 0$

$-36Ax^2 - 12Bx^2 = 0$

$-16Ax^3 = x^3 \rightarrow A = -\frac{1}{16}$

$\rightarrow +\frac{36}{16} = 12B \rightarrow B = \frac{3}{16}$

$\rightarrow -\frac{24}{16} - \frac{18}{16} \cdot 3 = -2C$

$C = \frac{24 + 54}{16 \cdot 2} = \frac{78}{32} = \frac{39}{16}$

$\rightarrow \frac{18}{16} - 6\frac{39}{16} - 4D = 0$

$D = \frac{\left(\frac{18}{16} - 6\frac{39}{16}\right)}{4}$

$y_{p_1} = -\frac{1}{16}x^4 + \frac{3}{16}x^3 + \frac{39}{16}x^2 + \frac{\left(\frac{18}{16} - 6\frac{39}{16}\right)}{4}x$

řešení $= y_h + y_{p_1} + y_{p_2}$

• Poté xe^x

→ Axe^x → nulová násobnost $\lambda = 1$ tedy to nerozšiřujeme x-kem

$y = Axe^x$

$y' = Ae^x + Axe^x = Ae^x(1+x)$

$y'' = Ae^x + Ae^x(1+x) = Ae^x(2+x)$

$y''' = Ae^x(3+x)$

→ získáváme

$Ae^x(3+x) - 3Ae^x(2+x) - 4Ae^x(1+x) = xe^x$

$A(3 + x - 6 - 3x - 4 - 4x) = x$

$A(3 - 6 - 4) = 0$ což je blbost

$A(x - 3x - 4x) = x$

→ $e^x(Ax + B)$ → $y' = Ae^x + e^x(Ax + B) = e^x(Ax + A + B)$

$y'' = e^x(Ax + 2A + B)$

$y''' = e^x(Ax + 3A + B)$

$\rightarrow (Ax + 3A + B) - 3(Ax + 2A + B) - 4(Ax + A + B) = x$

$\rightarrow Ax - 3Ax - 4Ax = x \rightarrow -6A = 1 \rightarrow A = -\frac{1}{6}$

$3A + B - 6A - 3B - 4A - 4B = 0 \rightarrow -7A = -1 \quad A = \frac{1}{7}$

$-7A - 6B = 0$

$y_{p_2} = e^x\left(\frac{1}{7}x - \frac{1}{6}\right)$

2019 – Početní část 1

ODR

triv. řešení $y=0$ … + není zde Lipschitzovsko (bkého)

① 7b

$$\frac{dy}{dx} = \frac{\sqrt{|y|}}{x^2+1} \longrightarrow \int \frac{1}{\sqrt{|y|}}\,dy = \int \frac{1}{x^2+1}\,dy$$

poč. podm. $y(1) = \frac{\pi^2}{64}$

$$\sqrt{|y|}\cdot 2 = \pm \operatorname{arctg} x + C$$

$$\frac{\partial \sqrt{|y|}}{\partial y} = \frac{1}{\sqrt{|y|}}\cdot\frac{1}{2}\cdot \operatorname{sign}|y|$$

$$|y| = \left(\frac{\pm \operatorname{arctg} x + C}{2}\right)^2$$

$$y = \pm\left(\frac{\pm \operatorname{arctg} x + C}{2}\right)^2 = \pm\frac{1}{4}\left(\pm \operatorname{arctg} x + C\right)^2$$

alt. řešení

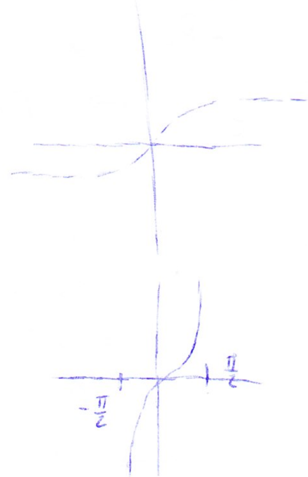
$$\int \frac{1}{\sqrt{|y|}}\,dy = \int \frac{1}{x^2+1}\,dx$$

a) $y>0$

$$\int \frac{1}{\sqrt{y}}\,dy = \int \frac{1}{x^2+1}\,dx \longrightarrow \sqrt{y}\cdot 2 = \operatorname{arctg} x + C_1$$

$$y = \frac{1}{4}\left(\operatorname{arctg} x + C_1\right)^2$$

$\operatorname{tg} x = \frac{\sin x}{\cos x}$



$\operatorname{tg} x = \frac{\sin x}{\cos x}$

b) $y<0$

$$y = -\frac{1}{4}\left(\operatorname{arctg} x + C_2\right)^2$$

$$\Longrightarrow \quad y(1) = \frac{\pi^2}{64} \longrightarrow \frac{\pi^2}{64} = +\frac{1}{4}\left(\operatorname{arctg} 1 + C_1\right)^2$$

$$= \frac{1}{4}\left(\frac{\pi}{4} + 0\right)^2 = \frac{\pi^2}{4\cdot 16} = \frac{\pi^2}{64}$$

$\longrightarrow y = \frac{1}{4}\operatorname{arctg}(x)^2$ díky poč. podm.

$\rightarrow$ pro $x \in (-\infty, k_1)$ $\quad y = -\frac{1}{4}\left(\operatorname{arctg}(x+k_1)\right)^2 \quad k_1 < 0$

to k_1 trochu jinak nepřed. V rámci auto c předním

$x \in (k_1, 0) \quad y = 0$

$x \in (0, \infty) \quad y = \frac{1}{4}\left(\operatorname{arctg}(x)\right)^2$

$+\exists$ navíc: řešení, které splňuje $y(-1) = -\frac{\pi^2}{64}$ jako $y(-1) = \frac{\pi^2}{64}$

existuje, pokud $k_1 = 0$

neexistuje. Pro $x<0$ máme y definované jako $y = \ominus\frac{1}{4}(\operatorname{arctg}(x)+c)^2$