Matematická analýza – cvičení VI

DÚ 1) $\boxed{\int \frac{x^3+1}{x^3-5x^2+6x}\,dx}$

Př: $\int \left(\frac{1-x}{x}\right)^2 dx = \int \frac{1-2x+x^2}{x^2}\,dx = \int \frac{1}{x^2}dx - 2\cdot\int\frac{1}{x}dx + \int 1\,dx$

$= -\frac{1}{x} - 2\ln|x| + x + C$

$\boxed{\int (a+b)\,dx = \int a\,dx + \int b\,dx}$

$\boxed{\int c\cdot f\,dx = c\cdot\int f\,dx}$

$\boxed{\int x^a\,dx = \frac{x^{a+1}}{a+1}}$ $\boxed{\int \frac{1}{x}\,dx = \ln|x|}$

• Všechny PF se liší pouze o konstantu, pouze když integrujeme na intervalu!

Př: $\int \frac{1}{x^2-x+2}\,dx = \int \frac{1}{(x-\frac{1}{2})^2+\frac{7}{4}}\,dx \left|\begin{matrix} x-\frac{1}{2}=u \\ dx=du \end{matrix}\right| = \int \frac{1}{u^2+\frac{7}{4}}\,du$

$= \frac{4}{7}\cdot\int \frac{1}{\frac{4}{7}u^2+1}\,du \left|\begin{matrix} z=\frac{2}{\sqrt{7}}u \\ dz=\frac{2}{\sqrt{7}}du \end{matrix}\right| = \frac{4}{7}\cdot\frac{\sqrt{7}}{2}\int \frac{1}{z^2+1}\,dz$

$= \frac{2}{\sqrt{7}}\operatorname{arctg}\left(\frac{2}{\sqrt{7}}\cdot\left(x-\frac{1}{2}\right)\right) + C = \frac{2}{\sqrt{7}}\cdot\operatorname{arctg}\left(\frac{2}{\sqrt{7}}\left(x-\frac{1}{2}\right)\right) + C$

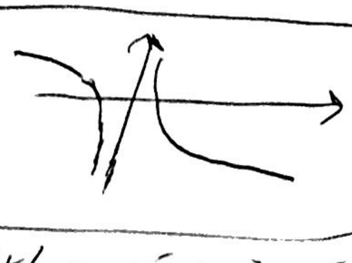
($f(g(x)) = \int f'(g(x))\cdot g'(x)\,dx$

$F'(x)=f(x)$ "$du=dx$" je pouze zkrácený zápis)

• Pokud integrujeme na sjednocení dvou intervalů, tak musíme psát:

$$F(x)\begin{cases} -\frac{1}{x} - 2\ln|x| + x + C_1 & x<0 \\ -\frac{1}{x} - 2\ln|x| + x + C_2 & x>0 \end{cases}$$

"lze je posouvat zvlášť"



• Naopak při integraci např. $|x|$ musíme dané PF "slepit".

Př: $\int tg^2 x\, dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int 1\, dx$

$= \underline{\underline{tg\, x - x}} + C$ (máme nekonečně mnoho intervalů → pro každý interval máme 1 konst.)

$x \in R - \{\frac{\pi}{2} + k\pi\}$

Př: $\int \frac{1}{1+\cos x} dx$

$= \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx = \int \frac{1}{\sin^2 x} dx - \int \frac{\cos x}{\sin^2 x} dx \quad \left| \begin{matrix} u = \sin(x) \\ du = \cos x\, dx \end{matrix} \right|$

$= -\cot g\, x - \int \frac{1}{u^2} du = \underline{\underline{-\cot g(x) + \frac{1}{\sin(x)}}} + C$ (opět nekonečně mnoho intervalů)

$\boxed{\int \frac{1}{\sin^2 x} = -\cot g(x)}$

Př: $\int x e^x dx = x e^x - \int e^x dx = \underline{\underline{x e^x - e^x + C}}$

$\left| \begin{matrix} u = x & v' = e^x \\ u' = 1 & v = e^x \end{matrix} \right|$

Př: $\int x^2 \sin(3x) dx = x^2 \cdot \left(-\frac{1}{3}\right) \cdot \cos 3x + \int \frac{2}{3} x \cos 3x =$

$\left| \begin{matrix} u = x^2 & v' = \sin 3x \\ u' = 2x & v = \frac{-\cos 3x}{3} \end{matrix} \right| \qquad \left| \begin{matrix} u = x & v' = \cos 3x \\ u' = 1 & v = \frac{\sin 3x}{3} \end{matrix} \right|$

$= -\frac{x^2}{3} \cdot \cos 3x + \frac{2}{3}\left(x \cdot \frac{\sin 3x}{3} - \int \frac{\sin 3x}{3} dx\right)$

$= -\frac{x^2}{3} \cos 3x + \frac{2}{9} x \sin 3x - \frac{2}{9} \cdot \int \sin 3x\, dx \quad \left| \begin{matrix} t = 3x \\ dt = 3dx \end{matrix} \right|$

$= \underline{\underline{-\frac{x^2}{3} \cos 3x + \frac{2}{9} x \sin(3x) + \frac{2}{27} \cos(3x) + C}}$

Př: $\int x\cdot \operatorname{arctg} x\,dx = \frac{x^2}{2}\cdot \operatorname{arctg} x - \int \frac{x^2}{2\cdot(1+x^2)}\,dx$

$\left| \begin{array}{ll} u = \operatorname{arctg} x & v' = x \\ u' = \frac{1}{1+x^2} & v = \frac{x^2}{2} \end{array} \right|$

$x^2 : x^2+1 = 1 - \frac{1}{x^2+1}$
$-(x^2+1)$

$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\cdot \int 1 - \frac{1}{x^2+1}\,dx$

$= \frac{x^2}{2}\cdot \operatorname{arctg} x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg} x + C$

Př: $\int \ln x\,dx = x\cdot \ln x - \int 1\,dx = x\cdot \ln x - x + C$

$\left| \begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{array} \right|$

Př: $\int e^{3x}\cos 2x\,dx = \left| \begin{array}{ll} v' = e^{3x} & v = \cos 2x \\ u = \frac{e^{3x}}{3} & v' = -2\sin 2x \end{array} \right| = \frac{e^{3x}}{3}\cos 2x + \frac{2}{3}\int e^{3x}\sin 2x\,dx$

$\left| \begin{array}{ll} u' = e^{3x} & v = \sin 2x \\ u = \frac{e^{3x}}{3} & v' = 2\cos 2x \end{array} \right| = \frac{e^{3x}}{3}\cos 2x + \frac{2}{3}\left(\frac{e^{3x}}{3}\sin 2x - \frac{2}{3}\int e^{3x}\cos 2x\,dx\right)$

=> rovnice:

$\left(1+\frac{4}{9}\right)\cdot \int e^{3x}\cos 2x\,dx = \frac{e^{3x}}{3}\cos 2x + \frac{2}{9}e^{3x}\sin 2x \quad \Big| \cdot \frac{9}{13}$

$\int e^{3x}\cdot \cos 2x\,dx = \frac{3e^{3x}}{13}\cos 2x + \frac{2}{13}e^{3x}\sin 2x$

$= \frac{3}{13}e^{3x}\cos 2x + \frac{2}{13}e^{3x}\sin 2x + C$

$= \frac{e^{3x}}{13}\cdot(3\cos 2x + 2\sin 2x) + C$

Př: $\int \cos^2 x\,dx =$ (použijeme $\cos^2 x = \frac{1+\cos 2x}{2}$)

$= \int \frac{1}{2}\,dx + \int \frac{\cos 2x}{2}\,dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$

$\left| \begin{array}{l} t = 2x \\ dt = 2dx \end{array} \right.$

NEBO: $\int \cos x\cdot \cos x\,dx = \cos x\cdot \sin x + \int \sin^2 x\,dx$! nelze znova per partes !

$\left| \begin{array}{ll} u = \cos x & v' = \cos x \\ u' = -\sin x & v = \sin x \end{array} \right| = \cos x\sin x + \int (1-\cos^2 x)\,dx$

$\Rightarrow \int \cos^2 x\,dx = \frac{1}{2}(\cos x\sin x + x) + C$

Př: $\int \sin^4 x\,dx = \int \sin^3 x \sin x\,dx = -\cos x \cdot \sin^3 x + \int \cos x\, 3\cdot \sin^2 x \cdot \cos x\,dx$

$= -\cos x \cdot \sin^3 x + 3\int (1-\sin^2 x)\sin^2 x\,dx$

$= -\cos x \cdot \sin^3 x + 3\cdot\int \sin^2 x\,dx - 3\cdot\int \sin^4 x\,dx$

$4\cdot\int \sin^4 x\,dx = -\cos x \sin^3 x + 3\cdot\int \sin^2 x$

- vyjádříme v závislosti na integrála s nižší mocninou

$\Rightarrow$ rekurentní vzorec

Př: $\int x e^{-x^2}dx = \left|\begin{matrix} t=-x^2 \\ dt=-2x\,dx\end{matrix}\right| = -\frac{1}{2}\cdot\int e^t\,dt = \underline{\underline{-\frac{1}{2}\cdot e^{-x^2}+C}}$

Př: $\int x^3\cdot a^{-x^2}dx = \left|\begin{matrix} -x^2=t \\ -2x\,dx=dt\end{matrix}\right| = \int a^t\cdot x^2\,\frac{-2x\,dx}{-2} = \int \frac{a^t\cdot(-t)\,dt}{-2}$

$= \int \frac{a^t\cdot t}{2}\,dt = \frac{1}{2}\cdot\int t\cdot a^t\,dt$ $\left|\begin{matrix} u=t & v'=a^t=e^{t\ln a} \\ u'=1 & v=\frac{a^t}{\ln a}\end{matrix}\right|$ $= \frac{1}{2}\cdot\left(t\cdot\frac{a^t}{\ln a} - \int\frac{a^t}{\ln a}\,dt\right) = \frac{1}{2}\cdot\left(t\cdot\frac{a^t}{\ln a} - \frac{a^t}{\ln^2 a}\right)+C$

$= \underline{\underline{\frac{a^t}{2\ln a}\cdot\left(t-\frac{1}{\ln a}\right)+C}} = \underline{\frac{a^{-x^2}}{2\ln a}\cdot\left(-x^2-\frac{1}{\ln a}\right)+C}$

$\int a^t = \frac{a^t}{\ln a}$

$= \underline{\underline{\frac{a^{-x^2}}{2\ln(a)}\cdot\left(-x^2-\frac{1}{\ln a}\right)+C}}$

Př: $\int\frac{1}{e^x+e^{-x}}dx = \int\frac{e^x}{e^{2x}+1}dx = \left|\begin{matrix} t=e^x \\ dt=e^x dx\end{matrix}\right| = \int\frac{1}{t^2+1}dt$

$= \underline{\operatorname{arctg}(e^x)+C}$ $\boxed{\text{• lze připsat } \cdot\frac{e^x}{e^x}}$

Př: $\int\frac{\ln^2 x}{x}dx = \left|\begin{matrix} t=\ln x \\ dt=\frac{1}{x}dx\end{matrix}\right| = \int t^2\,dt = \underline{\underline{\frac{t^3}{3}+C}}$

Př: $\int\frac{1}{\sqrt{1-x^2}\arcsin^2 x}dx = \left|\begin{matrix} u=\arcsin x \\ du=\frac{1}{\sqrt{1-x^2}}dx\end{matrix}\right| = \int\frac{1}{u^2}du = -\frac{1}{u}+C$

$= -\frac{1}{\arcsin(x)}+C$

Ř: $\int \sin^7 x\,dx = \int \sin^6 x \sin x\,dx = \int (\sin^2 x)^3 \sin x\,dx$

$= \int (1-\cos^2 x)^3 \sin x\,dx = -\int (1-t^2)^3\,dt$

$\left| \begin{matrix} t = \cos x \\ dt = -\sin x\,dx \end{matrix} \right| \quad = -\int (1-3t^2+3t^4-t^6)\,dt$

$= -\left(t - 3\frac{t^3}{3} + 3\frac{t^5}{5} - \frac{t^7}{7}\right) + C$

$= -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$

(přeskočíme #13)

Ř:

$\int \frac{x}{\cos^2 x}\,dx = x \cdot \mathrm{tg}\,x - \int \mathrm{tg}(x)\,dx = x\,\mathrm{tg}\,x + \int \frac{(-\sin x)}{\cos x}\,dx$

$\left| \begin{matrix} u = x & v' = \frac{1}{\cos^2 x} \\ u' = 1 & v = \mathrm{tg}\,x \end{matrix} \right| \qquad \left| \begin{matrix} t = \cos x \\ dt = -\sin x\,dx \end{matrix} \right|$

$= x \cdot \mathrm{tg}\,x + \int \frac{1}{t}\,dt = \underline{x \cdot \mathrm{tg}\,x + \ln|\cos x| + C}$

Ř: $\int \sin(\ln x)\,dx = \left| \begin{matrix} u = \ln x, \; x = e^u \\ du = \frac{1}{x}dx \Rightarrow du = \frac{1}{e^u}dx \end{matrix} \right|$

$= \int \sin(u) \cdot e^u\,du$ atd...

$= e^u \sin u - \int e^u \cos u\,du$

atd... $= \frac{e^u}{2}\sin u - \frac{e^u}{2}\cos u$

$= \frac{x}{2} \cdot \sin(\ln(x)) - \frac{x}{2}\cos(\ln(x))$

Matematická analýza - cvičení VII

$$\int \frac{x^3+1}{x^3-5x^2+6x}\,dx = x + \frac{1}{6}\ln|x| - \frac{9}{2}\ln|x-2| + \frac{28}{3}\ln|x-3| + C$$

maximální intervaly: $(-\infty,0), (0,2), (2,3), (3,+\infty)$

$$\frac{5x^2-6x+1}{x^3-5x^2+6x} = \frac{5x^2-6x+1}{x\cdot(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

"vynásobíme rovnici x", spočítáme $\lim_{x\to 0}$, "zakryju x a za x dosadím kořen"

(pokud je vícenásobný: lze takto pouze pro největší mocninu)

a) x musí být kořen

b) takto lze vypočítat pouze nejvyšší mocninu kořene

Př: $\int \frac{1}{(x^3+1)^2}\,dx$ — nesoudělné, — stupeň větší v ~~čitateli~~ jmenovateli } parciální zlomky

$x^3 = -1$, $x_1 = -1$ } $(x^3+1) = (x+1)(x^2-x+1)$

$x^3+1 : x+1 = x^2-x+1$
$-(x^3+x^2)$
$-x^2+1$
$-(-x^2-x)$
$x+1$

$$\frac{1}{(x^3+1)^2} = \frac{1}{(x+1)^2(x^2-x+1)^2} = \frac{A}{(x+1)^2} + \frac{B}{(x+1)} + \frac{Cx+D}{(x^2-x+1)^2} + \frac{Ex+F}{(x^2-x+1)}$$

a) $A = \frac{1}{9}$ ("zakrýváním")

b) upraví: $\frac{1}{(x+1)^2(x^2-x+1)^2} - \frac{\frac{1}{9}}{(x+1)^2} = \frac{1-\frac{1}{9}\cdot(x^2-x+1)^2}{(x+1)^2(x^2-x+1)^2} = \frac{1-\frac{1}{9}(\dots)}{(x+1)^2(x^2-x+1)^2}$

$(x^2-x+1)^2 = x^4-x^3+x^2-x^3+x^2-x+x^2-x+1 = x^4-2x^3+3x^2-2x+1$

$$= \frac{-\frac{1}{9}(x^4-2x^3+3x^2-2x-8)}{(x+1)^2(x^2-x+1)^2}$$

$(x^4-2x^3+3x^2-2x-8):(x+1) = x^3-3x^2+6x-8$
$-(x^4+x^3)$
$-3x^3+3x^2-2x-8$
$-(-3x^3-3x^2)$
$6x^2-2x-8$
$-(6x^2+6x)$
$-8x-8$

$$= \frac{-\frac{1}{9}(x^3-3x^2+6x-8)}{(x+1)(x^2-x+1)^2}$$

$\Rightarrow B = \frac{(-1-3-6-8)\cdot(-\frac{1}{9})}{9}$ ("zakrýváním")

$B = -2\cdot(-\frac{1}{9}) = \frac{2}{9}$

• pro C, D, E, F neznásobíme a dostaneme rovnici

$$1 = A\cdot(x^2-x+1)^2 + B(x^2-x+1)^2(x+1) + (Cx+D)(x+1)^2 + (Ex+F)(x+1)^2\cdot(x^2-x+1)$$

$$\left.\begin{array}{l} x^5(B+E) \\ x^4 \\ \vdots \\ A+D \end{array}\right\}\ \text{6 rovnic pro 6 neznámých}$$

$$\Rightarrow \quad \frac{3-2x}{9\cdot(x^2-x+1)} + \frac{1-x}{3(x^2-x+1)^2} + \underbrace{\frac{2}{9(x+1)} + \frac{1}{9(x+1)^2}}_{\text{jednoduché}}$$

$$\frac{2}{9}\int\frac{1}{x+1} = \frac{2}{9}\ln|x+1|$$

$$\frac{1}{9}\int\frac{1}{(x+1)^2} = -\frac{1}{9}\cdot\frac{1}{(x+1)}$$

$$\int\frac{3-2x}{9(x^2-x+1)}\,dx = -\frac{1}{9}\cdot\int\frac{2x-1-2}{x^2-x+1}$$

$$= -\frac{1}{9}\left(\int\frac{2x-1}{x^2-x+1}\,dx - 2\int\frac{1}{x^2-x+1}\,dx\right)$$

$$= -\frac{1}{9}\cdot\left(\ln|x^2-x+1| - 2\cdot\int\frac{1}{(x-\frac{1}{2})^2+1-\frac{1}{4}}\,dx\right)$$

$$= -\frac{1}{9}\cdot\left(\ln|x^2-x+1| - 2\cdot\int\frac{1}{\frac{3}{4}\cdot\left[\frac{4}{3}(x-\frac{1}{2})^2+1\right]}\,dx\right)$$

$$= -\frac{1}{9}\left(\ln|x^2-x+1| - \frac{2\cdot4}{3}\cdot\frac{\sqrt{3}}{2}\int\frac{1}{t^2+1}\,dt\right)$$

$$t = \frac{2}{\sqrt{3}}(x-\tfrac{1}{2}) \qquad dt = \frac{2}{\sqrt{3}}\,dx$$

$$= -\frac{1}{9}\left(\ln(x^2-x+1) - \frac{4\sqrt{3}}{3}\cdot\operatorname{arctg}\left(\frac{2}{\sqrt{3}}(x-\tfrac{1}{2})\right)\right) + C$$

• stejné pro

$$\int\frac{1-x}{3(x^2-x+1)^2} = -\frac{1}{6}\int\frac{2x+2}{(x^2-x+1)^2} = -\frac{1}{6}\int\frac{2x-1+2}{(x^2-x+1)^2} =$$

$$= -\frac{1}{6}\cdot\left(\int\frac{2x-1}{(x^2-x+1)^2}\,dx \overset{(-1)}{\cdot} \int\frac{1}{(x^2-x+1)^2}\,dx\right)$$

$$\int\frac{1}{(x^2-x+1)^2}\,dx \underset{\text{substituce}}{\Rightarrow} \int\frac{1}{(t^2+1)^2}\,dt$$

, trik: $\int\frac{1}{t^2+1}\,dx$ per partes

$$\left|\begin{array}{ll} u'=1 & v=(t^2+1)^{-1} \\ u=t & v'=(-1)\,2t\cdot(t^2+1)^{-2} \end{array}\right|$$

$$\int\frac{1}{t^2+1}\,dt = \frac{t}{t^2+1} + 2\int\frac{t^2}{(1+t^2)^2}\,dt = \frac{t}{t^2+1} + 2\int\frac{t^2+1}{(1+t^2)^2}\,dt - 2\int\frac{1}{(1+t^2)^2}$$

$$\boxed{\int\frac{1}{(1+t^2)^2}\,dt = \frac{1}{2}\cdot\left(\frac{t}{1+t^2} + \int\frac{1}{1+t^2}\,dt\right)}$$

3) $\int \frac{1}{x(1+2\sqrt{x}+\sqrt[3]{x})}\,dx = \left| \begin{matrix} t=\sqrt[6]{x} & 6t^5 = dx \\ t^6 = x & \end{matrix} \right| = \int \frac{6t^5}{t^6(1+2\cdot t^3+t^2)}\,dt$

$= 6 \cdot \int \frac{1}{t\cdot(t+1)(2t^2-t+1)}\,dt$

$2t^3+t^2+1 : (t+1) = 2t^2-t+1$
$-(2t^3+2t^2)$
$-t^2+1$
$-(-t^2-t)$
$t+1$

$= 3 \cdot \int \left(\frac{2}{t} - \frac{1}{2(t+1)} - \frac{6t-1}{4\cdot(t^2-\frac{t}{2}+\frac{1}{2})} \right) dt$

$= \ln|x| - \frac{3}{2}\ln(\sqrt[6]{x}+1) - \frac{9}{4}\ln\left(\sqrt[3]{x} - \frac{1}{2}\sqrt[6]{x} + \frac{1}{2}\right) + \frac{\sqrt{3}}{\sqrt{8}}\,\text{arctg}\left(\sqrt{\frac{8}{3}}\sqrt[6]{x} - \frac{1}{\sqrt{6}}\right)$

↑ speciální případ $\sqrt[s]{\frac{ax+b}{cx+d}}$ $\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{s}} = t$ substituce

$\Rightarrow x = \ldots$
$dx = \ldots$

4) přeskočíme

5) $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx = \left| \begin{matrix} x = \sin(t) \\ dx = \cos(t)\,dt \end{matrix} \right| = \int \frac{1\cdot\cos t}{(1-\sin^2 t)^{\frac{3}{2}}}\,dt = \int \frac{\cos t}{\cos^3 t}\,dt$

$= \int \frac{1}{\cos^2 t}\,dt = \text{tg}(t) + C = \text{tg}(\arcsin(x)) = \frac{\sin(\arcsin(x))}{\sqrt{1-\sin^2(\arcsin(x))}}$

$= \frac{x}{\sqrt{1-x^2}}$

6) $\int \frac{\sin^2(x)}{1+\sin^2(x)}\,dx = \left| \begin{matrix} t = \text{tg}(x) \\ \cancel{dt = \frac{1}{\cos^2 x}dx} \end{matrix} \right| \Rightarrow t^2 = \frac{\sin^2 x}{1-\sin^2 x} \Rightarrow \sin^2 x = \frac{t^2}{1+t^2}$

$x = \text{arctg}(t) \to dx = \frac{dt}{1+t^2}$

$= \int \frac{\frac{t^2}{1+t^2}}{1+\frac{t^2}{1+t^2}} \cdot \frac{1}{t^2+1}\,dt = \int \frac{t^2}{\cancel{1+t^2}} \cdot \frac{\cancel{1+t^2}}{1+2t^2} \cdot \frac{1}{t^2+1}\,dt = \int \frac{t^2}{(1+2t^2)(t^2+1)}\,dt$

$= \int \frac{1}{t^2+1}\,dt - \int \frac{1}{2t^2+1}\,dt = x - \frac{1}{\sqrt{2}}\,\text{arctg}(\sqrt{2}\,\text{tg}(x)) + C$

7) $\displaystyle\int \frac{1}{2\sin x - \cos x + 5}\,dx = \left| \begin{matrix} u = \operatorname{tg}\frac{x}{2} \\ x = 2\operatorname{arctg} u \end{matrix} \right| dx = \frac{2}{1+u^2}\,du \left| \begin{matrix} \cos x = \frac{1-u^2}{1+u^2} \\ \sin x = \frac{2u}{1+u^2} \end{matrix} \right|$

$$\cos x = \cos(2\operatorname{arctg} u) = \cos^2(\operatorname{arctg} u) - \sin^2(\operatorname{arctg}(u)) =$$
$$= \frac{1-\operatorname{tg}^2(\operatorname{arctg} u)}{1+\operatorname{tg}^2(\operatorname{arctg} u)} = \frac{1-u^2}{1+u^2}$$

$$\sin x = \sin(2\operatorname{arctg} u) = 2\cdot\underbrace{\sin(\operatorname{arctg} u)}_{\frac{u}{\sqrt{1+u^2}}}\,\underbrace{\cos(\operatorname{arctg} u)}_{\frac{1}{\sqrt{1+u^2}}} = \frac{2u}{1+u^2}$$

$$\int \frac{1}{2\cdot\frac{2u}{1+u^2} - \frac{1-u^2}{1+u^2} + 5}\cdot\frac{2}{1+u^2}\,du\ \cancel{dx} = \int \frac{1}{3u^2+2u+2}\,du = \int \frac{1}{\left(\sqrt{3}u+\frac{1}{\sqrt{3}}\right)^2+\frac{5}{3}}\,du$$

$$= \frac{3}{5}\int \frac{1}{\frac{3}{5}\left(\sqrt{3}u+\frac{1}{\sqrt{3}}\right)^2+1}\,du = \left| \begin{matrix} t = \frac{\sqrt{3}}{\sqrt{5}}\cdot\left(\sqrt{3}u+\frac{1}{\sqrt{3}}\right) \\ dt = \frac{3}{\sqrt{5}}\,du \end{matrix} \right|$$

$$= \frac{3}{5}\cdot\frac{\sqrt{5}}{3}\cdot\int\frac{1}{t^2+1}\,dt = \frac{\sqrt{5}}{5}\cdot\operatorname{arctg}\left(\frac{3u+1}{\sqrt{5}}\right)$$

$$= \frac{\sqrt{5}}{5}\operatorname{arctg}\left(\frac{3\cdot\operatorname{tg}\left(\frac{x}{2}\right)+1}{\sqrt{5}}\right)$$

• Eulerovy sub – NE v písemce