1) $f(x,y) = \ln(x+y)$ $\quad x+y>0$

$\frac{\partial f}{\partial x} = \frac{1}{x+y}$; $\frac{\partial f}{\partial y} = \frac{1}{x+y}$ $\quad x+y>0$

2) $f(x,y,z) = \cos x \cosh y$ $\quad (-\infty,\infty)\times(-\infty,\infty)\times(-\infty,\infty)$

$\frac{\partial f}{\partial x} = -\sin x \cosh y$

$\frac{\partial f}{\partial y} = \cos x \sinh y$

$\frac{\partial f}{\partial z} = 0$

3) $f(x,y) = |x|\,|y|$ $\quad (-\infty,\infty)\times(-\infty,\infty)$

• $x \neq 0$ $\quad \frac{\partial f}{\partial x} = \operatorname{sign} x\,|y|$

$y \neq 0$ $\quad \frac{\partial f}{\partial y} = \operatorname{sign} y\,|x|$

$x \to 0,\ y=0$ $\quad \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$

$y \neq 0\ x \to 0$ $\quad \lim_{x\to 0} \frac{f(x,y)-f(0,y)}{x} = \lim_{x\to 0} \frac{|x|\cdot|y|}{x} = \operatorname{sign} x\,|y|$

$\Rightarrow$ lim neexistuje

$y \to 0\ x \neq 0$ obdobně

4) $f(x,y) = \sqrt[3]{xy}$ $\quad \mathbb{R}\times\mathbb{R}$

$x \neq 0$ $\quad \frac{\partial f}{\partial x} = \frac{1}{3}\frac{y}{(xy)^{2/3}} = \frac{1}{3}\frac{y^{1/3}}{x^{2/3}}$

$y \neq 0$ $\quad \frac{\partial f}{\partial y} = \frac{1}{3}\frac{x^{1/3}}{y^{2/3}}$

$x \to 0,\ y=0$ $\quad \frac{\partial f}{\partial x}$ v $(0,0)$

$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$

$x \to 0\ y \neq 0$ $\quad \frac{\partial f}{\partial x}$ v $(0,y)$

$\lim_{x\to 0} \frac{f(x,y)-f(0,y)}{x} = \lim_{x\to 0} \frac{\sqrt[3]{xy}}{x}$ není $\in \mathbb{R}$

obdobně pro $y \to 0$ $\quad \frac{\partial f}{\partial y}$

(5) $f(x,y) = (x^5+y^5)^{1/5}$ $\quad \mathbb{R}\times\mathbb{R}$

$$\frac{\partial f}{\partial x} = \frac{5}{5}\frac{x^4}{(x^5+y^5)^{4/5}} \qquad x \neq -y$$

$$\frac{\partial f}{\partial y} = \frac{y^4}{(x^5+y^5)^{4/5}} \qquad x \neq y$$

v $(0,0)$ $\quad \frac{\partial f}{\partial x} = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 1$

$$\frac{\partial f}{\partial y} = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 1$$

(6) $f(x,y,z) = x^{\frac{y}{z}}$ $\quad (x,y,z)\in\mathbb{R}^3$ a

$x>0$ a $z\neq 0$

$x<0$ a $yz>0$ a $z\neq 0$

$$\frac{\partial f}{\partial x} = \frac{y}{z} x^{\frac{y}{z}-1} \qquad x>0 \text{ a } z\neq 0 \text{ nebo}$$

$x=0,\ z\neq 0 \quad \frac{y}{z}-1>0$

tohle jsi nějak zmotal, bacha

$$\frac{\partial f}{\partial y} \frac{\partial}{\partial y} e^{\frac{y}{z}\ln x} = \frac{\ln x}{z} x^{\frac{y}{z}} \qquad x>0,\ z\neq 0$$

$$\frac{\partial f}{\partial z} = -\frac{y}{z^2}\ln x\ x^{\frac{y}{z}} \qquad x>0,\ z\neq 0$$

e^((y/z)*lnx)

(7) $f(x,y) = (x^2+y^2)^\alpha \sin\frac{1}{x^2+y^2}$

Pro jaké hodnoty x bude mít funkce parc. derivace 1. řádu v bodě (0,0)? Tady to p. Běhounková řeší z definice - dosazení do toho def vzorečku

$$\frac{\partial f}{\partial x}(0,0) \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{x^{2\alpha}\sin\frac{1}{x^2}}{x} = \begin{cases} 0 & 2\alpha-1>0,\ \alpha>1/2 \\ \text{nelze} & \alpha = 1/2 \\ & \text{a } \alpha < 1/2 \end{cases}$$

u toho prvního by tam zůstala oscilující fce a u toho druhého bych dělil nulou

Asi kdy je parc. der. spojitá? Nepřečtu

2 kdy bude spojitá?

Tady klasicky derivuju.

$$\boxed{\frac{\partial f}{\partial x} = \alpha\cdot 2x(x^2+y^2)^{\alpha-1}\sin\frac{1}{x^2+y^2} - (x^2+y^2)^\alpha \underbrace{\cos\frac{1}{x^2+y^2}}_{\text{omezení}}\frac{2x}{(x^2+y^2)^2}}$$

u této části musí být alfa>1/2, myslím. Takže ta druhá podmínka je silnější.

chci aby v žádné ze dvou částí té derivace nebylo něco jako 1/x, abych nedělil nulou - pak by ta část divergovala, tedy by rostla nade všechny meze a v tom místě by fce nebyla spojitá

$2\alpha + 1 - 4 > 0 \rightarrow \alpha > 3/2$

pak $\lim_{x\to 0} \frac{\partial f}{\partial x} = 0$ **a parc. der. je spojitá**

(8) $f(x,y) = x^4+y^4-4x^2y^2$ $\quad f \in C^\infty$ na $\mathbb{R}$ → **záměnnost parc. derivací**

U následujících příkladů: spočtěte 2. parc. derivace, zjistěte jestli jsou záměnné

$\frac{\partial f}{\partial x} = 4x^3 - 8xy^2 \qquad \frac{\partial^2 f}{\partial y \partial x} = -16xy$

$\frac{\partial f}{\partial y} = 4y^3 - 8x^2y \qquad \frac{\partial^2 f}{\partial x \partial y} = -16xy$

(9) $f(x,y) = \frac{x}{y^2}$ $\quad y \neq 0$ na def. oboru je z C^2

$\frac{\partial f}{\partial x} = \frac{1}{y^2} \qquad \frac{\partial^2 f}{\partial y \partial x} = -\frac{2}{y^3}$

$\frac{\partial f}{\partial y} = \frac{-2x}{y^3} \qquad \frac{\partial^2 f}{\partial x \partial y} = -\frac{2}{y^3}$

(10) $f(x,y) = x\sin(x+y)$ $\quad \mathbb{R}\times\mathbb{R}$

$\frac{\partial f}{\partial x} = \sin(x+y) + x\cos(x+y)$

$\frac{\partial^2 f}{\partial y \partial x} = \cos(x+y) - x\sin(x+y)$

$\frac{\partial f}{\partial y} = x\cos(x+y)$

$\frac{\partial^2 f}{\partial x \partial y} = \cos(x+y) - x\sin(x+y)$

(11) $f(x,y) = \operatorname{tg}\frac{x^2}{y}$ $\quad y \neq 0 \quad \frac{x^2}{y} \neq k\pi + \frac{\pi}{2} \quad k \in \mathbb{Z}$

$\frac{\partial f}{\partial x} = \frac{2x}{y}\frac{1}{\cos^2\frac{x^2}{y}}$; $\frac{\partial^2 f}{\partial y \partial x} = -\frac{2x}{y^2}\frac{1}{\cos^2\frac{x^2}{y}} + \frac{2x}{y}\frac{+2\sin\frac{x^2}{y}}{\cos^3\frac{x^2}{y}}\cdot\left(-\frac{x^2}{y^2}\right) =$

$= -\frac{2x}{y^2}\frac{1}{\cos^2\frac{x^2}{y}}\left(1 - 2\frac{x^2}{y}\frac{\sin\frac{x^2}{y}}{\cos\frac{x^2}{y}}\right)$

$= -\frac{2x}{y^2}\frac{1}{\cos^2\frac{x^2}{y}}\left(1 - 2\frac{x^2}{y}\operatorname{tg}\frac{x^2}{y}\right)$

$\frac{\partial f}{\partial y} = -\frac{x^2}{y^2}\frac{1}{\cos^2\frac{x^2}{y}}$; $\frac{\partial^2 f}{\partial x \partial y} = -\frac{2x}{y^2}\frac{1}{\cos^2\frac{x^2}{y}} - \frac{x^2}{y^2}\frac{-2\sin\frac{x^2}{y}\cdot\frac{2x}{y}}{\cos^3\frac{x^2}{y}} =$

$= -\frac{2}{x^2}\frac{1}{\cos^2\frac{x^2}{y}}\left(1 - \frac{2x^2}{y}\operatorname{tg}\frac{x^2}{y}\right)$

(12) $f(x,y,z) = x^{y^z}$ $\quad x > 0$ a $y > 0$
$x = 0 \quad y > 0$
$x = 0 \quad y = 0 \quad z > 0$

$\frac{\partial f}{\partial x} = y^z x^{y^z - 1}$ $\quad x = 0 \quad y^z - 1 > 0$

$\frac{\partial f}{\partial y} \quad \frac{\partial}{\partial y} e^{y^z \ln x} = z \ln x\, y^{z-1} x^{y^z}$ $\quad x > 0$; $y = 0 \quad z - 1 > 0$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} e^{\ln x \cdot y^z} = \frac{\partial}{\partial z}\ln x\, y^z\, x^{y^z} = \ln x \ln y\, y^z x^{y^z}$ $\quad x, y > 0$

$\frac{\partial^2 f}{\partial y \partial x} = z y^{z-1} x^{y^z - 1} + y^z \frac{\partial}{\partial y} e^{(y^z - 1)\ln x} =$

$= z y^{z-1} x^{y^z - 1} + y^z z y^{z-1} \ln x\, x^{y^z - 1} =$

$= z y^{z-1} x^{y^z - 1}(1 + y^z \ln x)$

$\frac{\partial^2 f}{\partial x \partial y} = z \frac{1}{x} y^{z-1} x^{y^z} + z \ln x\, y^{z-1} y^z x^{y^z - 1} =$

$= z y^{z-1} x^{y^z - 1}(1 + \ln x\, y^{z})$

(13) $\operatorname{arctan} \frac{x+y}{1-xy}$ $xy \neq 1$

$\operatorname{arctan} z$ $z \in \mathbb{R}$ a $xy \neq 1$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} =$$

$$= \frac{1-xy+xy+y^2}{1-2xy+x^2y^2+x^2+y^2+2xy} = \frac{1+y^2}{1+x^2+y^2(1+x^2)} = \frac{1+y^2}{(1+x^2)(1+y^2)}$$

$$= \frac{1}{1+x^2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} \overset{\text{symetrie}}{=} \frac{1}{1+y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

(14) $$f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

záměnnost parc. derivací v $(0,0)$

? je funkce spojitá ?

Je fce spojitá? Zjistíme limitu as (x,y) approaches (0,0)

$$\boxed{\lim_{(x,y)\to(0,0)} xy\,\frac{x^2-y^2}{x^2+y^2} = 0 \quad \text{je spojitá}}$$

výpočet limity pomocí polární substituce. Šlo by i pomocí y=kx

$$0 \leq \left| xy\,\frac{x^2-y^2}{x^2+y^2} \right| = \frac{r^4}{r^2} \left| \cos x \sin x (\cos^2 x - \sin^2 x) \right| \leq 2r^2 \to 0$$

Derivace z definice v bodě (0,0)

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \lim_{x\to 0} \frac{\cancel{xy}\,\frac{x^2}{x^2}\Big|_{y=0} - 0}{x} = 0$$

$$\frac{\partial f}{\partial y}\Big|_{(0,0)} = \lim_{y\to 0} \frac{\cancel{xy}\,\frac{-y^2}{y^2} - 0}{y} = 0$$

$\frac{\partial f}{\partial x}$ pro $(x,y) \neq (0,0)$ **Derivace v okolí (0,0)**

$$\frac{\partial f}{\partial x} = y\,\frac{x^2-y^2}{x^2+y^2} + xy\,\frac{2x}{x^2+y^2} + xy(x^2-y^2)\frac{-2x}{(x^2+y^2)^2} =$$

$$= \frac{y}{x^2+y^2}\left(x^2-y^2+2x^2 - 2x^2\,\frac{(x^2-y^2)}{x^2+y^2}\right)$$

$$= \frac{y}{(x^2+y^2)^2}\left((x^2+y^2)(x^2-y^2) + 2x^2(x^2+y^2) - 2x^2(x^2-y^2)\right)$$

df/dx

$$= \frac{y}{(x^2+y^2)^2}\left(x^4 - y^4 + 4x^2y^2\right)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(xy\,\frac{x^2-y^2}{x^2+y^2}\right) =$$

$$= x\,\frac{x^2-y^2}{x^2+y^2} + \frac{xy}{x^2+y^2}(-2y) + xy\,\frac{(x^2-y^2)(-1)(2y)}{(x^2+y^2)^2} =$$

$$= \frac{x}{(x^2+y^2)^2}\Big((x^2-y^2)(x^2+y^2) - 2y^2(x^2+y^2) - 2y^2(x^2-y^2)\Big)$$

df/dy

$$= \frac{x}{(x^2+y^2)^2}\left(x^4-y^4-4x^2y^2\right)$$

$$\frac{\partial^2 f}{\partial x \partial y}\Big|_{0,0} = \lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} =$$

$$= \lim_{x\to 0} \frac{\frac{x}{x^4}(x^4) - 0}{x} = 1$$

Druhé derivace z definice. Liší se. Nejsou tedy záměnné.

$$\frac{\partial^2 f}{\partial y \partial x}\Big|_{00} = \lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y\,y^4}{y^4\,y} = -1$$

(15) x^2-y^2 v bodě $(1,1)$ ve směru jednotkového vektoru, který svírá s kladným směrem osy x úhel $\frac{\pi}{3}$

$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial x}(1,1) = 2 \qquad$ nebo $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} = 2$

$\frac{\partial f}{\partial y} = -2y \qquad \frac{\partial f}{\partial y}(1,1) = -2 \qquad \lim_{h\to 0} \frac{(y+h)^2-y^2}{h} = -2$

f je C^∞ a tedy

$v = \begin{pmatrix} \cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \qquad \frac{\partial f}{\partial v}(a) = \nabla f(a)\cdot v$

$v\cdot v = \frac{1}{4} + \frac{3}{4} = 1$ ✓

$$\frac{\partial f}{\partial v}\Big|_{(1,1)} = \nabla f \cdot v = \begin{pmatrix} 2 \\ -2 \end{pmatrix}^T \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = 1-\sqrt{3}$$

nebo přímo z definice

$$\frac{\partial f}{\partial v}(a) = \lim_{h\to 0} \frac{f(a+vh)-f(a)}{h}$$

$$\frac{\partial f}{\partial v}((1,1)) = \lim_{h\to 0} \frac{(1+\frac{1}{2}h)^2 - (1+\frac{\sqrt{3}}{2}h)^2 - 0}{h} = 1-\sqrt{3}$$

(16) $f(x,y) = x^2 - xy + y^2$ v bodě $(1,1)$

jednotkový vektor s nejmenší, největší a nulovou derivací

největší nárůst $v = \frac{\nabla f(a)}{\|\nabla f(a)\|}$ $\to$ $-v$ největší pokles

-v největší pokles

$\frac{\partial f}{\partial x} = 2x - y$ $\quad \frac{\partial f}{\partial x}((1,1)) = 1$

$\frac{\partial f}{\partial y} = -x + 2y$ $\quad \frac{\partial f}{\partial y}((1,1)) = 1$

$v_{MAX} = \frac{1}{\sqrt{2}}(1,1)$ největší růst

$v_{MIN} = -\frac{1}{\sqrt{2}}(1,1)$ největší pokles

$v_{ZERO}^{1} = \frac{1}{\sqrt{2}}(1,-1)$

$v_{zero}^{2} = \frac{1}{\sqrt{2}}(-1,1)$

(17) ? $\frac{\partial F}{\partial u}$ $\quad F = f(g),\ f(x,y,z)$

$g_1(u,v) = \frac{u^2-1}{2v}$

$g_2(u,v) = \frac{u+v}{u-v}$ $\quad u \neq v$

$g_3(u,v) = u^2 v^2$

$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$

$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$

$\frac{\partial g_1}{\partial u} = \frac{2u}{2v} = \frac{u}{v}$

$\frac{\partial g_2}{\partial u} = \frac{(u-v) - (u+v)}{(u-v)^2} = \frac{-2v}{(u-v)^2}$

$\frac{\partial g_3}{\partial u} = 2u$

$\frac{\partial g_1}{\partial v} = -\frac{u}{v^2}$

$\frac{\partial g_2}{\partial v} = \frac{(u-v)+(u+v)}{(u-v)^2} = \frac{2u}{(u-v)^2}$

$\frac{\partial g_3}{\partial v} = -2v$

$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\frac{u}{v} + \frac{\partial f}{\partial y}\left(\frac{-2v}{(u-v)^2}\right) + \frac{\partial f}{\partial z}(2u)$

(18) $f(s,t)$ hladká, nezáporná na $\mathbb{R}^2$

$g(x,y) = f(x,y)^{f(y,x)}$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} e^{f(y,x)\ln f(x,y)} = f(x,y)^{f(y,x)} \left(\ln f(x,y) \frac{\partial f}{\partial t}\Big|_{(y,x)} + \frac{f(y,x)}{f(x,y)} \frac{\partial f}{\partial s}\Big|_{(x,y)} \right)$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y} e^{f(y,x)\ln f(x,y)} = f(x,y)^{f(y,x)} \left(\ln f(x,y) \frac{\partial f}{\partial s}\Big|_{(y,x)} + \frac{f(y,x)}{f(x,y)} \frac{\partial f}{\partial t}\Big|_{(x,y)} \right)$$