(1) metoda postupných aproximací

$y' = ay \quad y(0) = \varkappa$

→ řešení

$y' - ay = 0 \quad e^{-ax}$

$(y e^{-ax})' = 0 \quad y = c e^{ax} \quad y(0) = \varkappa$

$\to c = \varkappa$

$y = \varkappa e^{ax}$

- metoda postupných aproximací

$y' = ay \quad \longrightarrow \int_0^x$

$y(x) = \varkappa + a \int_0^x y(x')\,dx'$

$T(y) \overset{def}{=} \varkappa + a \int_0^x y(x')\,dx'$

a $y_{n+1} = T(y_n)$

$y_0 \overset{def}{=} 0$

$y_1 = T(y_0) = \varkappa$

$y_2 = T(y_1) = \varkappa + a \int_0^x \varkappa\,dx' = \varkappa + \varkappa ax = \varkappa(1 + ax)$

$y_3 = T(y_2) = \varkappa + a \int_0^x \varkappa(1 + ax')\,dx' = \varkappa\left(1 + ax + \frac{a^2x^2}{2}\right)$

$\vdots$

$y_{n+1} = \varkappa\left(1 + ax + \frac{a^2x^2}{2} + \dots + \frac{a^nx^n}{n!}\right)$

$y_n \xrightarrow{n\to\infty} \varkappa e^{ax}$

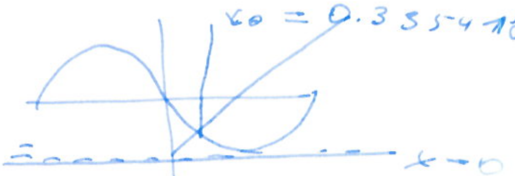
Zkoumejme $\rho(T(y_1), T(y_2))$, $y \in C([0,x])$

$(P, \rho) = (C([0,x]), \max)$

$\rho(T(y_1), T(y_2)) = \sup_{x \in (0,x)} \left| a \int_0^x (y_1(x') - y_2(x'))\,dx' \right| = |a| \sup_{x \in (0,x)} |y_1(x) - y_2(x)|\, X$

$= |a| x \rho(y_1, y_2) \qquad |a| x < 1$ pak T je kontrahující zobrazení → $\exists!$ řešení + další prodlužování $\underbrace{\frac{1}{|a| + \varepsilon}}_{\text{známe}}$, $\varepsilon > 0$

② přibližné řešení $2x + \sin x = 1$



$$2x + \sin x = 1 \to x = \frac{1-\sin x}{2}$$

$$T(x) = \frac{1-\sin x}{2}$$ → má-li pevný bod x_0 → $x_0 = \frac{1-\sin x_0}{2}$ je řešením

$$x_{n+1} = T(x_n)$$

$$x_0 = 0$$

$$x_1 = \frac{1}{2}$$

$$x_2 = \frac{1-\sin 1/2}{2}$$

$$\vdots$$

$$x_{12} = 0.33538$$

$$|T(x_1) - T(x_2)| = \frac{1}{2}\,|\sin(x_1) - \sin(x_2)| = \frac{1}{2}\,|\cos\xi\,(x_1 - x_2)| \le \frac{1}{2}\,|x_1 - x_2|$$

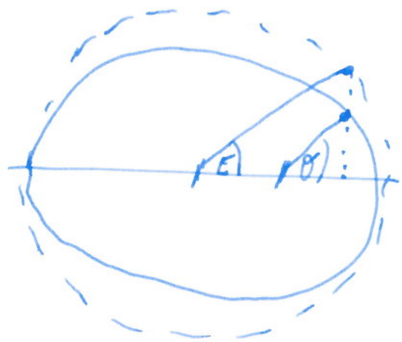
$$\to \rho(T(x_1), T(x_2)) \le \frac{1}{2}\,\rho(x_1, x_2)$$ → T kontrakce → $\exists!\ x_0 \in \mathbb{R}$

• příklad Keplerova rovnice

$E - e\sin E = n t = M$ – střední anomálie

$$\tan\frac{\vartheta}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}$$

$$r = \frac{a(1-e^2)}{1+e\cos\vartheta}$$



③ $y(x) = \frac{1}{2}\int_0^1 x^2 s\, y(s)\, ds + x$

- analyticky $\quad y(x) = \alpha x^2 + x$

$y(x) = \alpha x^2 + x = \frac{1}{2}\int_0^1 x^2 s(\alpha s^2 + s)\, ds + x$

$= \frac{1}{2} x^2 \left[\alpha \frac{s^4}{4} + \frac{s^3}{3}\right]_0^1 + x =$

$= \frac{1}{2}\left(\frac{\alpha}{4} + \frac{1}{3}\right) x^2 + x$

$\Rightarrow \alpha = \frac{1}{2}\left(\frac{\alpha}{4} + \frac{1}{3}\right).$

$24\alpha = 3\alpha + 4 \rightarrow \qquad \alpha = \frac{4}{21} = 0.190476$

- $T(y) = \frac{1}{2}\int_0^1 x^2 s\, y(s)\, ds + x$

$y_0(x) = 0$

$y_1(x) = x$

$y_2(x) = \frac{1}{2}\int_0^1 x s \cdot s\, ds + x = \frac{1}{2} x \left[\frac{s^3}{3}\right]_0^1 + x = \underset{0.1666}{\frac{1}{6}} x^2 + x$

$y_3(x) = \frac{1}{2}\int_0^1 x^2 s\left(\frac{1}{6}s^2 + s\right) + x = \frac{1}{2} x^2 \left[\frac{1}{6}\frac{s^4}{4} + \frac{s^3}{3}\right]_0^1 + x =$

$= \frac{3}{16} x^2 + x$

$\quad 0.1875 \qquad \alpha_3 = \frac{3}{16}$

$y_4(x) : \alpha_4 x^2 + x \qquad \alpha_4 = \frac{1}{2}\left(\frac{\alpha_3}{4} + \frac{1}{3}\right) = 0.19014$

$\alpha_5 = \frac{1}{2}\left(\frac{\alpha_4}{4} + \frac{1}{3}\right) = 0.190429$

$\alpha_6 \qquad = 0.190470$

$\vdots$

? $\exists!$? $\quad \sup_{\langle 0,1\rangle} |T(y_1) - T(y_2)| = \rho(T(y_1), T(y_2))$

$= \sup \left|-\frac{1}{6}x^2 - x + x\right| = \sup \frac{1}{6}x^2 = \frac{1}{6} < 1$

(4) $x_{n+1} = x_n - \frac{1}{2}(x_n^2 - a) \longrightarrow \sqrt{a}$ (iterační metoda)

$x_0 = 0$

$T(x) = x - \frac{1}{2}(x^2 - a)$ $\quad x_n \to \tilde{x} \to$ musí platit

$T(\tilde{x}) = \tilde{x} \to \tilde{x} = \tilde{x} - \frac{1}{2}(\tilde{x}^2 - a) \Leftrightarrow \tilde{x} = \sqrt{a}$

$T(x) = x - \frac{1}{2}x^2 + \frac{a}{2} = x\left(1 - \frac{x}{2}\right) + \frac{a}{2}$ na $\langle 0, \sqrt{a} \rangle$

$x_0 = 0$

$x_1 = \frac{a}{2}$

$x_2 = \frac{a}{2} - \frac{1}{2}\frac{a^2}{4} + \frac{a}{2} = a\left(1 - \frac{a}{8}\right)$

$\to$?rostoucí a omezená $\to \sqrt{a}$

$\to x_n^2 - a < 0$ a $x_{n+1} \geq x_n$

pomocí B.V.

$T(x) \quad x\left(1 - \frac{x}{2}\right) + \frac{a}{2}$ $\quad 0 = T'(x): 1 - \frac{2x}{2} = 0 \quad x = 1$

nelklesající na $\langle 0, 1 \rangle$

max $T(x)$ na $\langle 0, \sqrt{a} \rangle = T(\sqrt{a})$

$T(\sqrt{a}) = \sqrt{a}\left(1 - \frac{\sqrt{a}}{2}\right) + \frac{a}{2} = \sqrt{a}$

$T(x) \leq \sqrt{a}$ pro $x \in \langle 0, \sqrt{a} \rangle$

$|T(x_1) - T(x_2)| = \left|x_1 - x_2 - \frac{x_1^2 - x_2^2}{2}\right| = \left|(x_1 - x_2)\left(1 - \frac{x_1 + x_2}{2}\right)\right|$

$= |x_1 - x_2| \left|1 - \frac{x_1 + x_2}{2}\right| \leq |x_1 - x_2| (1 - a)$

$x_1, x_2 \geq \frac{a}{2}$ $\quad 0 < a \leq 1$ $\quad T$ kontrakce

Limity fcí více proměnných

$\lim_{(x,y)\to(0,0)} (x+y+1)$ $\qquad \mathbb{R}^2 \to \mathbb{R}$

$\forall \varepsilon > 0 \ \exists \delta \ \forall (x,y) \quad 0 < \|(x,y) - (a,y)\| < \delta \Rightarrow$

$\Rightarrow |f(x,y) - L| < \varepsilon$

$\|(x,y)\| \qquad P_\delta(0,0)$

$|(x,y)-(0,0)| = |x-0| + |y-0| < \delta$

$\Rightarrow x+y+1 \in U_\varepsilon(1)$

$|x+y+1-1| < \varepsilon$

$|x+y| \leq |x| + |y| < \delta = \varepsilon$ $\qquad \forall \varepsilon$ stačí zvolit $\delta = \varepsilon$

(5) $\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2}$

$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = \left| \begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix} \right| = $

$= \lim_{r\to 0^+} r^{2\, r^4 \sin^2\varphi \cos^2\varphi} = \lim_{r\to 0^+} e^{r^4 \cos^2\varphi \sin^2\varphi \ln r^2} =$

$= e^{\lim_{r\to 0^+} r \ln r^2 \; \underbrace{r^3}_{\to 0} \; \underbrace{\cos^2\varphi \sin^2\varphi}_{\text{konečné}}} = 1$

$\lim_{r\to 0} r \ln r^2 = \lim_{r\to 0} \frac{\ln r}{\frac{1}{r}} \overset{l'H}{=} \lim_{r\to 0} \frac{\frac{1}{r}}{-\frac{1}{r^2}} = 0$

lim r*lnr=0 —> e^(0)=1. Takže řešení je 1. Mám to tak i v sešitě.

(6) $\lim_{\|(x,y)\|\to\infty} \frac{\overset{\sim r}{x+y}}{\underset{\sim r^2}{x^2-xy+y^2}}$ $\rightarrow$ podezíráme 0 (ale rozorna $\ominus$ x 0)

$= \lim_{\|\,\|\to\infty} \frac{x+y}{\frac{1}{2}(x^2+y^2+(x-y)^2)}$ $\rightarrow$ podezíráme 0

dva policajti

$0 \leq \lim_{\|\,\|\to\infty} \left| \frac{x+y}{x^2-xy+y^2} \right| \leq \lim_{\|(x,y)\|\to\infty} \frac{|x|+|y|}{\frac{1}{2}(x^2+y^2+\underbrace{(x-y)^2}_{\geq 0})} \leq$

$\leq \lim_{\to\infty} \frac{|x|+|y|}{\frac{1}{2}(x^2+y^2)} = \left| \begin{matrix} (x-y)^2 \geq 0 \\ x^2+y^2 \geq +2xy \\ 2x^2+2y^2 \geq x^2+y^2+2xy \end{matrix} \right| \leq \lim \frac{\sqrt{x^2+y^2}}{x^2+y^2} =$

$\sqrt{x^2+y^2} \geq \frac{x+y}{2}$

~~[illegible]~~

$= \lim \frac{1}{\sqrt{x^2+y^2}} = 0$ $\qquad$ norm

jde také řešit polární substitucí - sešit

⑦ $\lim\limits_{\|(x,y)\| \to \infty} \frac{x^2+y^2}{x^4+y^4}$ $\sim r^2$ $\sim r^4$ čekáme 0

$$\lim_{r \to \infty} \frac{r^2}{r^4(\cos^4\varphi + \sin^4\varphi)} = \lim_{r\to\infty} \frac{1}{r^2(\cos^4\varphi+\sin^4\varphi)}$$

→ musíme ukázat, že $\cos^4\varphi + \sin^4\varphi \neq 0$

→ najít vyjádření stačilo

$(\cos^4\varphi + \sin^4\varphi)' = 4\cos\varphi\sin\varphi(\underbrace{-\cos^2\varphi}_{1-\sin^2\varphi} + \underbrace{\sin^2\varphi}_{1-\cos^2\varphi})$

$2\sin^2\varphi - 1$ → stacionární body $(0, \pi, 3/2\pi, \pi/2)$

$\cos^4 0 + \sin^4 0 = 1$ $\in [1/2, 1]$

$\cos^4\pi + \sin^4\pi = 1$

$\cos^4\frac{\pi}{2} + \sin^4\frac{\pi}{2} = \frac{1}{2}$

→ nebo ~~$\cos^2\varphi \in [0,1]$ → $\cos^2\varphi \geq \cos^4\varphi$~~
~~$\sin^2\varphi \in [0,1]$ → $\sin^2\varphi \geq \sin^4\varphi$~~

$\cos^4\varphi + \sin^4\varphi = \cos^4\varphi + 2\sin^2\varphi\cos^2\varphi + \sin^4\varphi - 2\sin^2\varphi\cos^2\varphi =$

$$= \underbrace{(\cos^2\varphi + \sin^2\varphi)^2}_{1} - 2\sin^2\varphi\cos^2\varphi = \underbrace{1 - \underbrace{\frac{1}{2}\underbrace{\sin^2 2\varphi}_{[0,1]}}_{\in[0,1/2]}}_{\in[1/2,1]}$$

$$0 \leq \left| \frac{1}{r^2(\cos^4\varphi + \sin^4\varphi)} \right| \leq \frac{2}{r^2} \to 0$$

(8) $\lim\limits_{(x,y)\to(0,a)} \frac{\sin xy}{x} \overset{\uparrow}{=} \lim\limits_{(x,y)\to(0,a)} \frac{\sin xy}{xy}\, y = a$

$a \neq 0$
(spojité složení)

$\lim\limits_{(x,y)\to(0,a)} \left(\frac{\sin xy}{x}\right) \overset{\uparrow}{=} 0$

$a = 0$
blížíme se $x \to 0$
$y \to 0$
ale $x \neq 0, y \neq 0$

(9) $\lim\limits_{(x,y)\to(0,0)} \frac{x^6+y^6}{x^2-y^2}$ pozor $|x| \neq |y|$ (pro tuto variantu nemá smysl)

→ budeme zkoušet křivky?

- $x = 0 \quad y \to 0$

$\lim\limits_{y\to 0} \frac{y^6}{-y^2} = 0$

- $x = f(y)$

x^2 chci aby se dostalo jako y^2 → $x = y + \dots$ a zároveň aby $(y+y^p)^2$ se dostalo nějaké y^6 po odečtení y^2

$y\,y^p = y^6 \to p = 5$

$x = y + y^5$

$\lim\limits_{(x,y)\to 0} \frac{x^6+y^6}{x^2-y^2} = \lim\limits_{y\to 0} \frac{y^6+y^6+o(y^6)}{y^2+2y^6+y^{10}-y^2} =$

$= \lim\limits_{y\to 0} \frac{2y^6+o(y^6)}{2y^6+o(y^6)} = 1$ → limity po křivkách se neshodují

(10) $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy \sim r^2}{x^2+y^2 \sim r^2}$ přechodu do polárních souřadnic

limita neexistuje. Lze dokázat pomocí polární substituce - zbude lim as r->0 2sin(fí)cos(fí). Tedy v závislosti na úhlu se to liší.

- $x = y$ $\quad \lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = \lim\limits_{x\to 0} \dfrac{2x^2}{2x^2} = 1$

$x = 0$ $\quad \lim\limits_{y\to 0} \dfrac{\overbrace{2xy}^{0}}{x^2+y^2} = 0$

limita neexistuje

další

$\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{3x^2+3y^2} = \dfrac{1}{3}\lim\limits_{x,y\to(0,0)} \dfrac{x^2+y^2 - \frac{(x^2+y^2)^3}{3!} + \dots}{x^2+y^2} = \dfrac{1}{3}$

$xy \le \dfrac{x^2+y^2}{2}$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$ $\quad 0 \le \left|\dfrac{xy^2}{x^2+y^2}\right| \le \left|\dfrac{\frac{x^2+y^2}{2}y}{x^2+y^2}\right| = \dfrac{|y|}{2} \le$

$\dfrac{1}{2}\sqrt{x^2+y^2}$ $\quad \varphi = 0$

young $\quad xy \le \dfrac{x^p}{p} + \dfrac{y^q}{q} \quad \dfrac{1}{p}+\dfrac{1}{q}=1$

$x,y>0$

(11) ~~lim~~ $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2}$

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right)=\lim_{x\to 0}0=0$$

$$\lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right)=\lim_{y\to 0}0=0$$

$$\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}\begin{cases} x=y & \lim_{x\to y}\frac{x^4}{x^4}=1 \\ x=0 & \lim_{y\to 0}0=0 \end{cases}\Bigg\}\ \text{limita neexistuje}$$

(12) ~~lim~~ $f(x,y)=(x+y)\sin\frac{1}{x}\sin\frac{1}{y}$

$$\lim_{y\to 0}f(x,y)=\lim_{y\to 0}\underbrace{(x+y)}_{\to 0}\sin\frac{1}{x}\underbrace{\sin\frac{1}{y}}_{}=\lim_{y\to 0}x\sin\frac{1}{x}\sin\frac{1}{y}\to\text{neexistuje}$$

$$\lim_{(x,y)\to(0,0)}(x+y)\underbrace{\sin\frac{1}{x}\sin\frac{1}{y}}_{\text{omezená}}$$

$$0\le|x+y|\left|\sin\frac{1}{x}\right|\left|\sin\frac{1}{y}\right|\le|x+y|\to 0$$

další příklady

$$\lim \frac{x^2+\sin^2 y}{(x+y)^2}$$

$y=0 \quad \lim_{x\to 0} \frac{x^2}{x^2}=1$

$x=\sin y \quad \lim \frac{2\sin^2 y}{(2\sin y)^2}=\frac{1}{2}$

$\to$ lim neexistuje

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)}{|x|+|y|}$$

$(x,y)\in P_\delta(0,0) \quad x\neq 0, y\neq 0$

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+xy)}{xy}\, xy$$

$xy\to 0$

$\frac{\ln(1+xy)}{xy}$ omezená na $P_\delta(0,0)$

nebo

$\lim_{\xi\to 0} \frac{\ln(1+\xi)}{\xi}=1$

$$0\leq \left|\frac{\ln(1+xy)}{|x|+|y|}\right| \leq K\frac{|x||y|}{|x|+|y|} \leq \tilde{K} r \to 0$$

$x=0, y\neq 0$ nebo $y=0, x\neq 0$

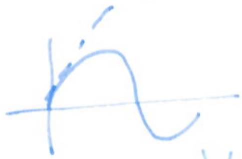
$$\frac{\ln(1+xy)}{|x|+|y|}=0$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y - y^3}{x^2+y^2} = 0$$

$x = r\cos\varphi$
$y = r\sin\varphi$
полярни к.

$$0 \le \left| \frac{3x^2y - y^3}{x^2+y^2} \right| \le r \left| \frac{3\cos^2\varphi\sin\varphi - \sin^3\varphi}{1} \right| \le 4r \to 0$$

ако $(x,y)\to(0,0)$



$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^3+y^4)}{x^2+y^2} = 0$$

$$0 \le \left| \frac{\sin(r^3\cos^3\varphi + r^4\sin^4\varphi)}{r^2} \right| \le \frac{1}{r^2} \left| r^3\cos^3\varphi + r^4\sin^4\varphi \right| \le 2r \to 0$$

$$\lim_{(x,y)\to(0,0)} \frac{e^{-\frac{1}{x^2}}y}{e^{-\frac{2}{x^2}}+y^2}$$

$y=0$ $\quad \lim_{x\to 0} \frac{0}{x^2} = 0$

$y = e^{-1/x^2}$ $\quad \lim_{x\to 0} \frac{e^{-\frac{2}{x^2}}}{e^{-\frac{2}{x^2}}+e^{-\frac{2}{x^2}}} = \frac{1}{2}$

} lim несъществува

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2} = 1$$

$xy \le \frac{1}{2}(x^2+y^2)$ $\quad$ (т.к. $(x-y)^2 \ge 0$)

$0 \le x^2y^2 \le \frac{1}{4}(x^2+y^2)^2 \le x^2+y^2$ ако $0 < x^2+y^2 < 1$

$1 = (x^2+y^2)^0 \ge (x^2+y^2)^{x^2y^2} \ge (x^2+y^2)^{x^2+y^2}$

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)^{(x^2+y^2)} = \lim_{r\to 0^+} r^r = 1$$