Zjistěte, kde má fce totální diferenciál. Určete ho. (př. 1 až 6)

(1) $f(x,y)=\ln(x+y)$ $\qquad x+y>0$

$\frac{\partial f}{\partial x}=\frac{1}{x+y}$ $\qquad x \neq -y$

$\frac{\partial f}{\partial y}=\frac{1}{x+y}$

"splňuje postačující podmínky existence tot diferenciálu (spojité parc derivace na jistém okolí v bodech (x,y) přísluší R^2, x+y>0)"

$df(x,y)(h_1,h_2)=\frac{1}{x+y}h_1+\frac{1}{x+y}h_2=(h_1+h_2)\frac{1}{x+y}$

→ splňuje postačující podmínky existence tot. diferenciálu (spoj. parc. derivace na jistém okolí) v bodech $(x,y)\in \mathbb{R}^2$, $x+y>0$

(2) $f(x,y,z)=\cos x\cosh y$ $\qquad (x,y,z)\in\mathbb{R}^3$

→ **spojité parc. derivace**

$\frac{\partial f}{\partial x}=-\sin x\cosh y$ $\qquad df(x,y,z)(h_1,h_2,h_3)=$

$\frac{\partial f}{\partial y}=\cos x\sinh y$ $\qquad -\sin x\cosh y\,h_1+\cos x\sinh y\,h_2$

$\frac{\partial f}{\partial z}=0$

(3) $f(x,y)=|x||y|$ $\qquad (x,y)\in\mathbb{R}^2$, spojitá

→ tot. diferenciál existuje mimo osy $x\neq0$ a $y\neq0$

tot. diferenciál existuje pouze mimo osy, tedy x a y nesmí být 0

pro $x\neq0$ $\quad \frac{\partial f}{\partial x}=\operatorname{sign}x\,|y|$

pro $y\neq0$ $\quad \frac{\partial f}{\partial y}=\operatorname{sign}y\,|x|$

$df(x,y)(h_1,h_2)=\operatorname{sign}x\,|y|\,h_1+\operatorname{sign}y\,|x|\,h_2$

• pro $x=0$ $\quad \lim_{x\to0}\frac{|x||y|}{x}$ existuje der. pouze pro $y=0$

Víceméně derivace z definice (místo x je tedy jen h). Ale stačí se podívat na ty dvě derivace nahoře a i z toho je vidět, že v bodě (0,0) to dává smysl.

• pro $y=0$ obdobně

? ∃ tot. diferenciál v $(0,0)$

tj. je splněno $\lim_{h\to0}\frac{f(x+h_1,y+h_2)-f(x,y)-df(x,y)(h_1,h_2)}{\|h\|}=0$

tj. je splněno (předpis definice tot. dif.)

pro $(x,y)=(0,0)$ $\qquad$ AG nerovnost

$0\le\lim_{h\to0}\frac{|h_1||h_2|-0-0}{\|h\|}\le\lim_{h\to0}\frac{\frac12(h_1^2+h_2^2)}{\|h\|}=\lim_{h\to0}\frac12\frac{\|h\|^2}{\|h\|}=0$

→ v bodě $(0,0)$ $\quad df(0,0)(h_1,h_2)=0$

④ $f(x,y) = \sqrt[3]{xy}\,;\ (x,y) \in \mathbb{R}^2$

$x \neq 0 \quad \frac{\partial f}{\partial x} = \frac{1}{3}\frac{\sqrt[3]{xy}}{x}$

$y \neq 0 \quad \frac{\partial f}{\partial x} = \frac{1}{3}\frac{\sqrt[3]{xy}}{y}$

pro $x \neq 0$ a $y \neq 0$ $\quad df(x,y)(h_1,h_2) = \frac{1}{3}\sqrt[3]{xy}\left(\frac{h_1}{x} + \frac{h_2}{y}\right)$

pro $x = 0$

$\frac{\partial f}{\partial x}\Big|_{(0,y)} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{hy}}{h} = \begin{cases} 0 & \text{pro } y = 0 \\ \text{neex} & y \neq 0 \end{cases}$

$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$

$\frac{\partial f}{\partial y}$ rovnako $\Rightarrow$

$\exists ? \lim_{h \to 0} \frac{f(0+h_1, 0+h_2) - f(0,0) - L\cdot h}{\|h\|} = 0$

$= \lim_{h\to 0} \frac{\sqrt[3]{h_1 h_2} - 0 - 0}{\|h\|} = \lim_{h \to 0} \frac{h^{2/3}}{2|h|} = \infty$

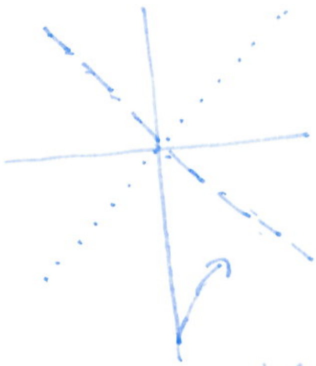
$\rightarrow$ tot. dif. neexistuje

napríklad pro $h_1 = h_2 = h$

⑤ $f(x,y) = \sqrt[5]{x^5 + y^5}$

$x = -y \quad \frac{\partial f}{\partial x} = \frac{x^4}{(x^5+y^5)^{4/5}}$

$\frac{\partial f}{\partial y} = \frac{y^4}{(x^5+y^5)^{4/5}}$



neni diferencovatelná

$df(x,y)(h_1,h_2) = \frac{1}{(x^5+y^5)^{4/5}}(h_1x^4 + h_2y^4)$ pro $x \neq -y$.

co v $x = y = 0$?

$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{x}{x} = 1$

$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{y}{y} = 1$

L, který by mohlo byť tot. diferenciál

$L = h_1 + h_2$

$\exists \lim_{h\to 0} \frac{f(h_1,h_2) - f(0,0) - h_1 - h_2}{\|h\|} \overset{?}{=} 0$

$\lim_{h \to 0} \frac{\sqrt[5]{h_1^5 + h_2^5} - h_1 - h_2}{\|h\|} = \lim_{h \to 0} \frac{\sqrt[5]{h_1^5+h_2^5} - h_1 - h_2}{\sqrt[5]{h_1^5 + h_2^5}} =$

můžeme libovolné $\|\ \|$

$= \lim_{h\to 0} \frac{\sqrt[5]{h_1^5+h_2^5}}{\sqrt[5]{|h_1|^5+|h_2|^5}} - \frac{h_1+h_2}{\sqrt[5]{|h_1|^5 + h_2^5}} = \lim_{h\to 0} \frac{\sqrt[5]{h_1^5+h_2^5}}{\sqrt[5]{|h_1|^5+|h_2|^5}} - \frac{h_1+h_2}{\sqrt[5]{|h_1|^5+|h_2|^5}} =$

např. pro $h_1 = h_2 = h$

$\lim_{h \to 0} \mathrm{sign}\, h - \frac{2h}{\sqrt[5]{2}\,|h|} \neq 0 \rightarrow$ tot. diferenciál v $(0,0)$ neexistuje

(6) $f(x,y,z) = x^{\frac{y}{z}}$ $\quad x>0$ a $z\neq 0$

$\qquad x<0$ a $\frac{y}{z}>0$ $z\neq 0$

$\frac{\partial f}{\partial x} = \frac{y}{z} x^{\frac{y}{z}-1}$ $\quad x>0$ $z\neq 0$

$\qquad x<0$ $z\neq 0$ $\frac{y}{z}>1$

$\frac{\partial f}{\partial y} = \frac{\ln x}{z} x^{\frac{y}{z}}$ $\quad x>0$ $z\neq 0$

$\frac{\partial f}{\partial z} = -\frac{y}{z^2}\ln x\, x^{\frac{y}{z}}$ $\quad x>0,\ z\neq 0$

pro $x>0$ a $z\neq 0$

$df(x,y,z)(h_1,h_2,h_3) = x^{\frac{y}{z}}\left(\frac{y}{zx}h_1 + \frac{\ln x}{z}h_2 - \frac{y}{z^2}\ln x\, h_3\right)$

(7) $f(x,y) = (x^2+y^2)^{\alpha}\sin\frac{1}{x^2+y^2}$ $\quad \alpha\in\mathbb{R}$

? tot. diferenciál 1. řádu v 0,0

$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{x^{2\alpha}\sin\frac{1}{x^2}}{x} =$ $\quad 2\alpha-1>0 \quad \alpha>\frac{1}{2}$

$= \lim_{x\to 0}\left(\frac{x^{2\alpha}}{x}\right)\left(\sin\frac{1}{x^2}\right) \overset{\alpha>\frac{1}{2}}{=} 0$

omezená

→ potřebujeme aby → 0

aby $\frac{\partial f}{\partial x}$ v $(0,0)$ existovala

$\frac{\partial f}{\partial y}(0,0)$ obdobně

$\exists ? \lim_{h\to 0}\frac{f(h_1,h_2)-f(0,0)-0}{\|h\|} \overset{?}{=} 0$

$\lim_{h\to 0}\frac{(h_1^2+h_2^2)^{\alpha}-0-0}{\|h\|} = \lim_{h\to 0}\frac{\|h\|^{2\alpha}}{\|h\|} \to 0$ pro $2\alpha-1>0$

→ pozor spojitost parc. derivací není zaručena

(8) $\underline{f(x,y,z)}$ spojitě má tot. diferenciál

$$\left.\begin{array}{ll} x = u^2+v^2 & = g_1 \\ y = u^2-v^2 & = g_2 \\ z = 2uv & = g_3 \end{array}\right] \in C^1(\mathbb{R}^2) \rightarrow \text{mají tot. diferenciál}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} =$$

$$= \frac{\partial f}{\partial x}\Big|_{x(u,v),y(u,v),z(u,v)} \cdot 2u + \frac{\partial f}{\partial y}\Big|_{\dots} 2u + \frac{\partial f}{\partial z}\Big|_{\dots} 2v$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\Big|_{\dots} 2v - \frac{\partial f}{\partial y}\Big|_{\dots} 2v + \frac{\partial f}{\partial z}\Big|_{\dots} 2u$$

$$df(u,v)(h_1,h_2) = h_1\left(2u\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)+2v\frac{\partial f}{\partial z}\right) + h_2\left(2v\left(\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\right)+2u\frac{\partial f}{\partial z}\right)$$

(9) f má tot. diferenciál v $(1,1)$

$$g(s,u) = f(f(u,s), f(s,u))$$

$$\frac{\partial g}{\partial x_1}(1,1) = ? \quad , f(1,1) = 1$$

$$g(x,y) = f(x,y) \qquad x = f(u,s) \qquad y = f(s,u)$$

$$\frac{\partial f}{\partial x_1}(1,1) = 1$$

$$\frac{\partial f}{\partial x_2}(1,1) = 2$$

$$\frac{\partial g}{\partial x_1} = \frac{\partial f}{\partial x}\Big|_{f(u,s),f(s,u)} \frac{\partial f(u,s)}{\partial s} + \frac{\partial f}{\partial y}\Big|_{f(u,s),f(s,u)} \frac{\partial f(s,u)}{\partial s}$$

$$= \frac{\partial f}{\partial x}(1,1)\frac{\partial f}{\partial s}(1,1) + \frac{\partial f}{\partial y}(1,1)\frac{\partial f}{\partial x}(1,1) = 1\cdot 2 + 2\cdot 1 = 4 \; ???$$

(10) $d^3 f(x,y,z)$ $\quad f(x,y,z)=xyz$

$$d^2 f(a)(h)=\sum_{i,j=1}^{N} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)h_i h_j$$

$$d^m f(a)(h)=\sum_{i_1,\dots,i_m}^{N} \frac{\partial^m f}{\partial x_{i_1}\dots\partial x_{i_m}}(a)h_{i_1}\dots h_{i_m}$$

$\frac{\partial f}{\partial x}=yz \quad \frac{\partial f}{\partial y}=xz \quad \frac{\partial f}{\partial z}=xy$

$df = yzh_1 + xzh_2 + xyh_3$

$d^2 f = (zh_2+yh_3)h_1 + (zh_1+xh_3)h_2 + (yh_1+xh_2)h_3$

$d^3 f = 2h_1h_2h_3 + 2h_1h_2h_3 + 2h_1h_2h_3 = 6h_1h_2h_3$

nebo také

$\frac{\partial f}{\partial x}=yz \quad \frac{\partial^2 f}{\partial x^2}=0 \quad \to$ nemohou 3. derivace $\frac{\partial^3 f}{\partial x\partial y\partial z}$

$\frac{\partial^2 f}{\partial y\partial x}=z \quad \frac{\partial^2 f}{\partial z\partial x}=y \dots$

(11) a) $1.02^2\cdot 2.003^3\cdot 3.004^3 = f(1,2,3)+df(1,2,3)(0.02,0.003,0.004)$

$f = x^2y^3z^3$

$df(a)(h_1,h_2,h_3) = 2xy^3z^3h_1 + 3x^2y^2z^3h_2 + 3x^2y^3z^2h_3$

$f(1,2,3)=1^2\cdot \underbrace{2^3}_{8}\cdot \underbrace{3^3}_{27} = 216$

$df(1,2,3)(0.02,0.003,0.004) =$

$= 2\cdot 1\cdot 2^3\cdot 3^3\cdot \frac{2}{100} + 3\cdot 1\cdot 2^2\cdot 3^3\cdot\frac{3}{1000} + 3\cdot 1^2\cdot 2^3\cdot 3^2\frac{4}{1000} =$

$= \frac{97\cdot 27}{250}$ $\quad$ celkem $216+\frac{97\cdot 27}{250} \doteq 226.476$

přesně 226.643

b) $\sin 29^\circ \cdot \operatorname{tg} 46^\circ = f\left(\frac{\pi}{6},\frac{\pi}{4}\right) + df\left(\frac{\pi}{6},\frac{\pi}{4}\right)\left(-\frac{\pi}{180},\frac{\pi}{180}\right)$

$f(x,y)=\sin x \operatorname{tg} y$

$\frac{\partial f}{\partial x}=\cos x \operatorname{tg} y \quad \frac{\partial f}{\partial y}=\frac{\sin x}{\cos^2 y}$

$\frac{\partial f}{\partial x}\Big|_{\frac{\pi}{6},\frac{\pi}{4}} = \frac{\sqrt3}{2}\cdot 1 \quad \frac{\partial f}{\partial y}\Big|_{\frac{\pi}{6},\frac{\pi}{4}} = \frac{\frac12}{\left(\frac{\sqrt2}{2}\right)^2}=1$

$df\left(\frac{\pi}{6},\frac{\pi}{4}\right)\left(-\frac{\pi}{180},\frac{\pi}{180}\right) = -\frac{\sqrt3}{2}\frac{\pi}{180}+1\cdot\frac{\pi}{180} = 0.002338 = \frac{\pi}{180}\left(1-\frac{\sqrt3}{2}\right)$

celkem 0.502338

přesně 0.502035