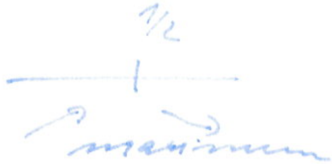
(1) xy ; $x+y=1 \to y=1-x$

$f(x,y) = x(1-x)$

$\frac{\partial f}{\partial x} = 1-2x = 0 \quad x = \frac{1}{2}$

$\Rightarrow y = 1/2$



$F(x,y,\lambda) = xy - \lambda(x+y-1)$

$\frac{\partial F}{\partial x} = y - \lambda \doteq 0$

$\frac{\partial F}{\partial y} = x - \lambda \doteq 0$

$\frac{\partial F}{\partial \lambda} =$ vazba $= -(x+y-1) \doteq 0$

$y = \lambda = x$

$\Rightarrow x = \frac{1}{2}$ a $y = 1/2 \to \lambda = 1/2$

$\left(\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y} \right) = (1 \quad 1)$ $h=1$ ok

(2) $f(x,y) = \frac{x}{a} + \frac{y}{b}$; $x^2+y^2=1$

$g = x^2+y^2-1 = 0$

$\frac{\partial g}{\partial x} = 2x$

$\frac{\partial g}{\partial y} = 2y$

protože musí být $x=0, y=0$ zároveň $h=1$

$\to$ ale $(0,0)$ nesplňuje vazební podmínku

$F(x,y,\lambda) = \frac{x}{a} + \frac{y}{b} - \lambda(x^2+y^2-1)$

$\frac{\partial F}{\partial x} = \frac{1}{a} - 2\lambda x \doteq 0 \quad \to \frac{1}{a^2} = 4\lambda^2 x^2$

$\frac{\partial F}{\partial y} = \frac{1}{b} - 2\lambda y \doteq 0 \quad \frac{1}{b^2} = 4\lambda^2 y^2$

$\frac{\partial F}{\partial \lambda} = -(x^2+y^2-1) \doteq 0$

$\frac{1}{a^2} + \frac{1}{b^2} = 4\lambda^2(\underbrace{x^2+y^2}_{1\ \text{z vazby}})$

$\to \lambda = \pm \frac{1}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$

$\to \lambda = \pm \frac{1}{2}\frac{\sqrt{a^2+b^2}}{ab}$

$x = \frac{1}{2a\lambda} = \pm \frac{b}{\sqrt{a^2+b^2}}$ P_+

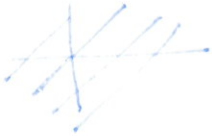
$y = \frac{1}{2b\lambda} = \pm \frac{a}{\sqrt{a^2+b^2}}$ P_-

$f(x^\pm, y^\pm) = \pm \frac{1}{\sqrt{a^2+b^2}}\left(\frac{b}{a} + \frac{a}{b}\right)$

$a \cdot b > 0$ P_+ je maximum P_- minimum

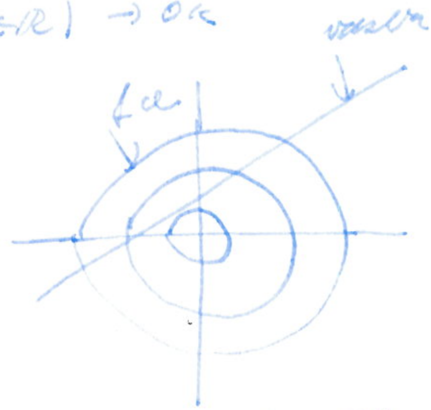
$a \cdot b < 0$ P_- je maximum P_+ minimum

③ $f(x,y) = x^2+y^2$ $\qquad \frac{x}{a}+\frac{y}{b}=1$



$g = \frac{x}{a}+\frac{y}{b}-1=0$

$\nabla g = \left(\frac{1}{a}, \frac{1}{b}\right)$ hodnost 1 $(a,b\in\mathbb{R}) \to$ ok



$F(x,y,\lambda) = x^2+y^2-\lambda\left(\frac{x}{a}+\frac{y}{b}-1\right)$

$\frac{\partial F}{\partial x} = 2x - \lambda\frac{1}{a} \stackrel{!}{=} 0$

$\frac{\partial F}{\partial y} = 2y - \lambda\frac{1}{b} \stackrel{!}{=} 0$

$\frac{\partial F}{\partial \lambda} = -\left(\frac{x}{a}+\frac{y}{b}-1\right) \stackrel{!}{=} 0$

$\lim\limits_{\|x\|\to\infty} f(x) = \infty$

f spojitá

$\Rightarrow$ musí nabývat minima

$$\left.\begin{aligned} 2x &= \frac{\lambda}{a} \\ 2y &= \frac{\lambda}{b} \\ \frac{x}{a}+\frac{y}{b} &= 1 \end{aligned}\right\} \quad \begin{aligned} &\frac{\lambda}{2a^2}+\frac{\lambda}{2b^2}=1 \\ &\lambda\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=2 \\ &\lambda = \frac{2a^2b^2}{a^2+b^2} \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} x &= \frac{a\cdot b^2}{a^2+b^2} \\ y &= \frac{a^2 b}{a^2+b^2} \end{aligned}\right\} \text{minimum}$$

Var. II. – alternativa (lineární vazba, jde dosadit)

$x^2+y^2 \qquad bx+ay=ab$

$x = \frac{ab-ay}{b} = a - \frac{a}{b}y$

$\tilde{f}(y) = x^2+y^2 = a^2 - 2\frac{a^2}{b}y + \frac{a^2}{b^2}y^2 + y^2$

$\frac{\partial \tilde{f}}{\partial y} = -2\frac{a^2}{b} + 2\left(\frac{a^2}{b^2}+1\right)y = 0$

$y = \frac{a^2b}{a^2+b^2}$

$\to x = (\text{zpětně}) = \frac{a\,b^2}{a^2+b^2}$

4 $x^m y^n z^p$, $x+y+z=a$, $m,n,p,a>0$

$g(x,y,z)=x+y+z-a=0$

$\nabla g = (1,1,1)$ $h=1$ → splněny předpoklady.

$F(x,y,z,\lambda)=x^m y^n z^p-\lambda(x+y+z-a)$

$\frac{\partial F}{\partial x}= m x^{m-1} y^n z^p-\lambda \doteq 0 \rightarrow \lambda = m x^{m-1} y^n z^p \;/x$

$\frac{\partial F}{\partial y}= n x^m y^{n-1} z^p-\lambda \doteq 0 \rightarrow \lambda = n x^m y^{n-1} z^p \;/y$

$\frac{\partial F}{\partial z}= p x^m y^n z^{p-1}-\lambda \doteq 0 \rightarrow \lambda = p x^m y^n z^{p-1} \;/z$

$-(x+y+z-a) \doteq 0$

$\Sigma: x^m y^n z^p(m+n+p)$
$= \lambda \underbrace{(x+y+z)}_{\text{rovno } a} = \lambda a$

dále
$m x^{m-1} y^n z^p = n x^m y^{n-1} z^p = p x^m y^n z^{p-1}$ =p*x^m*y^n*z^(p-1)

$\Rightarrow \frac{m}{n}=\frac{x}{y}$ $\frac{n}{p}=\frac{y}{z}$

$\lambda = m x^{m-1} y^n z^p$
$x^m y^n z^p(m+n+p)=\lambda a = a m x^{m-1} y^n z^p$
$\frac{x^m}{x^{m-1}} = x = \frac{am}{m+n+p}$

podobně $y=\frac{na}{m+n+p}$

$z=\frac{ap}{m+n+p}$

$\lim_{|x|\to\infty} f(x)=\infty$
→ má minimum
$(x,y,z)=\frac{a}{m+n+p}(m,n,p)$

→ ověření: můžeme složitě vyšetřovat i Hessovu matici (viz kapitola)

$$
\begin{aligned}
d^2 f = \; & m(m-1)\, x^{m-2} y^{n} z^{p} (dx)^2 \\
& n(n-1)\, x^{m} y^{n-2} z^{p} (dy)^2 \\
& p(p-1)\, x^{m} y^{n} z^{p-2} (dz)^2 \\
& 2mn\, x^{m-1} y^{n-1} z^{p}\, dx\, dy \\
& 2mp\, x^{m-1} y^{n} z^{p-1}\, dx\, dz \\
& 2np\, x^{m} y^{n-1} z^{p-1}\, dy\, dz
\end{aligned}
$$

z vazby $dx + dy + dz = 0 \quad dz = -dx - dy$
(z vnitřní linear. dif. (ověřování))

$$
\begin{aligned}
x^{m-2} y^{n-2} z^{p-2} \Big(& m(m-1)\, y^2 z^2\, dx^2 \\
& n(n-1)\, x^2 z^2\, dy^2 \\
& p(p-1)\, x^2 y^2 (dx^2 + 2\,dx\,dy + dy^2) \\
& 2mn\, xyz^2\, dx\, dy \\
& 2mp\, xy^2 z\, dx(-dx - dy) \\
& 2np\, x^2 yz\, dy(-dx - dy) \Big)
\end{aligned}
$$

→ a dále upravit matici

(5) $f(x,y,z) = \sin x \sin y \sin z$ $\underline{x,y,z > 0}$

$x+y+z = \frac{\pi}{2}$

$g = x+y+z-\frac{\pi}{2}$

$\nabla g = k(1,1,1) \; - \; k = 1$ ok

$L = F(x,y,z,\lambda) = \sin x \sin y \sin z - \lambda\left(x+y+z-\frac{\pi}{2}\right)$

$$\left.\begin{aligned}
\frac{\partial F}{\partial x} &= \cos x \sin y \sin z - \lambda \overset{!}{=} 0 \\
\frac{\partial F}{\partial y} &= \sin x \cos y \sin z - \lambda \overset{!}{=} 0 \\
\frac{\partial F}{\partial z} &= \sin x \sin y \cos z - \lambda \overset{!}{=} 0 \\
\frac{\partial F}{\partial \lambda} &= -\left(x+y+z-\frac{\pi}{2}\right) \overset{!}{=} 0
\end{aligned}\right\} \Rightarrow \begin{aligned}
&\cos x \sin y = \sin x \cos y \\
&\quad (\text{wo } \sin z \neq 0) \\
&\cos y \sin z = \sin y \cos z \\
&\quad (\text{wo } \sin x \neq 0)
\end{aligned}$$

$$\frac{\operatorname{tg} y}{\operatorname{tg} x} = \frac{\operatorname{tg} z}{\operatorname{tg} y} = 1 \Rightarrow \underbrace{x}_{+k\pi} = \underbrace{y}_{+l\pi} = \underbrace{z}_{+m\pi} \rightarrow x = y = z = \frac{\pi}{6}$$

$\lim\limits_{\|(x,y)\| \to \infty} f(x) = ?? \; \nexists$

Hessian f:

$\frac{\partial^2 f}{\partial x^2} = -\sin x \sin y \sin z$ $\frac{\partial^2 f}{\partial x \partial y} = \cos x \cos y \sin z$

$\frac{\partial^2 f}{\partial y^2} = -\sin x \sin y \sin z$ $\frac{\partial^2 f}{\partial x \partial z} = \cos x \sin y \cos z$

$\frac{\partial^2 f}{\partial z^2} = -\sin x \sin y \sin z$ $\frac{\partial^2 f}{\partial y \partial z} = \sin x \cos y \cos z$

$\sin\frac{\pi}{6} = \frac{1}{2}$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Hessian

$$\begin{pmatrix} -\frac{1}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & -\frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & -\frac{1}{8} \end{pmatrix} = \frac{1}{8}\begin{pmatrix} -1 & 3 & 3 \\ 3 & -1 & 3 \\ 3 & 3 & -1 \end{pmatrix}$$

ainda $\frac{1}{8}$

$-dx^2 - dy^2 - dz^2 + 6(dx\,dy + dy\,dz + dx\,dz)$

$$= \underbrace{-(dx + 3dy + 3dz)^2}_{-dx^2 - 9dy^2 - 9dz^2 + 6dx\,dy + 6dz\,dy + 6dz\,dy} + 8dy^2 + 8dz^2 \rightarrow \text{indefinit}$$

→ zkusíme fikovat směr

$dx = -dy - dz$

$$-(\boxed{dy^2}+2dy\,dz+dz^2)\ \boxed{-dy^2}-dz^2-\underline{6((-dy-dz)dy\ (-dy-dz)dz\ dy\,dz} =$$

$$= -8dy^2-8dz^2-8dy\,dz = -8(dy^2+dz^2+dy\,dz)$$

$$= -8\left(\frac{1}{2}(dy+dz)^2+\frac{1}{2}dy^2+\frac{1}{2}dz^2\right)$$ → negativní definitní ⇒ maximum

(6) $\sum_{i=1}^{n} x_i^p \quad \sum_{i=1}^{n} x_i = a \quad p > 1,\ a \geq 0$

$g = \sum x_i - a$

$\nabla g_k = (\underbrace{1, \dots, 1}_{n})$ ok

$F(\vec{x}, \lambda) = \sum_{i=1}^{n} (x_i^p - \lambda x_i) + \lambda a$

$\frac{\partial F}{\partial x_j} = $ (nenulová souřadnice) $= p x_j^{p-1} - \lambda \stackrel{!}{=} 0$

$\frac{\partial F}{\partial \lambda} = $ vazba $\rightarrow \sum_{i=1}^{n} x_i = a$

$\sum_{j=1}^{n} \frac{\partial F}{\partial x_i} x_i = \sum_{i=1}^{n} p x_i^p - \lambda \sum_{i=1}^{n} x_i \overset{\text{vazba}}{=} \sum_{i=1}^{n} p x_i^p - \lambda a = 0$

navíc $\forall j \quad \lambda = p(x_i)^{p-1} \quad \forall i = 1 \dots n$

$\Rightarrow x_i = x_j \quad \forall i, j \quad i \neq j$

$n p x_i^p = \lambda a \rightarrow \lambda = \frac{np}{a} x_i^p$

$p(x_i)^{p-1} = \frac{np}{a} x_i^p$

$\Rightarrow \boxed{x_i = \frac{a}{n}} \quad \forall i = 1, \dots, n$

$\frac{\partial^2 L}{\partial x_i^2} = p(p-1) x_i^{p-2}$

$\frac{\partial^2 L}{\partial x_i \partial x_j} = 0$

hessián $\begin{pmatrix} p(p-1) & & \\ & \ddots & \\ & & p(p-1) \end{pmatrix} x_i^{p-2}$

$a \geq 0$
$p > 1$

$= \begin{pmatrix} p(p-1) & & \\ & \ddots & \\ & & p(p-1) \end{pmatrix} \left(\frac{a}{n}\right)^{p-2}$

pro $a \neq 0$ pozitivně definitní $\rightarrow$ minimum

(7) $x - 2y - 3$ na C

$0 \le x \le 1$

$0 \le y \le 1$

$0 \le x + y \le 1$

omezená i uzavřená, musí dosáhnout minimum i maximum

(0) vnitřek oblasti

$\underbrace{\frac{\partial f}{\partial x} = 1 \qquad \frac{\partial f}{\partial y} = -2}$

$\to$ nemůže se nacházet extrém

(1) $y = 0$ a $0 \le x \le 1$

$f(x,0) = x - 3$ $\to$ extrémy mohou nabývat v krajních bodech

$f(0,0) = -3$

$f(1,0) = -2$

(3) $x = 0$ a $0 \le y \le 1$

$f(0,y) = -2y - 3$

$f(0,0) = -3$

$f(0,1) = -5$

(3) $x + y = 1 \to x = 1 - y$

$f(1-y, y) = 1 - y - 2y - 3 = -3y - 2 \longrightarrow y = 1 \quad f(x,y) = -5$

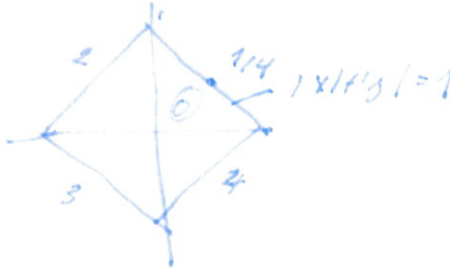
$y = 0 \quad f(x,y) = -2$

$\Rightarrow$ MAX v $(1,0)$ $\quad f(x,y) = -2$

$\Rightarrow$ MIN v $(0,1)$ $\quad f(0,1) = -5$

(8) $x^2 - xy + y^2$; $|x| + |y| \le 1$

eliptický paraboloid



možnosti $\frac{\partial f}{\partial x} = 2x - y \stackrel{!}{=} 0$ $\quad y = 2x \}$ $x = y = 0$

$\frac{\partial f}{\partial y} = 2y - x \stackrel{!}{=} 0$ $\quad x = 2y \}$ $f(0,0) = 0$

(1) $x + y \le 1$ $\quad (x,y) = (1,0) \to (x,y) = (0,1)$

$g = x + y - 1 = 0$ $\quad \nabla g = (1,1)$ ok

$x = 1 - y$

$(1-y)^2 - (1-y)y + y^2 = 3y^2 - 3y + 1$

$f'(y) = 6y - 3 = 0$

$y = \frac{1}{2}$ ok

$f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$

(3) ze symetrie $\quad f(-\frac{1}{2}, -\frac{1}{2}) = \frac{1}{4}$

$-x - y = 1$

$\cancel{x} = -1 - y$

(2) $-x + y = 1$ $\quad (y-1)^2 - (y-1)y + y^2 =$

$y = 1 + x$ $\quad = y^2 - 2y + 1 - y^2 + y + y^2 = y^2 - y + 1$

$x = y - 1$

$y = \frac{1}{2}$

$x = -\frac{1}{2}$

$f(-\frac{1}{2}, \frac{1}{2}) = \frac{3}{4} \to$ MAX

(4) symetrie

$(x,y) = (\frac{1}{2}, -\frac{1}{2}) \to \frac{3}{4} \to$ MAX

→ funkce na ohraničené uzavřené, i spojitá → maximum i minimum

→ MAX $\frac{3}{4}$ v $(-1/2, 1/2)$ a $(1/2, -1/2)$

MIN 0 v $(0,0)$

(9) $x^2+y^2-12x+16y$. $x^2+y^2 \le 25$

uvnitř

$\nabla f = (2x-12, 2y+16) \doteq (0,0)$

$\left.\begin{array}{l} x=6 \\ y=-8 \end{array}\right\}$ nesplňuje $x^2+y^2 \le 25$

na kružnici

$F(x,y,\lambda) = x^2+y^2-12x+16y-\lambda(x^2+y^2-25)$

$g = x^2+y^2-25=0$

$\nabla g = (2x, 2y)$ pro $(x,y)=(0,0)$ $\lambda=0$, ale neleží na kružnice

pro $(x,y) \neq (0,0)$

$\nabla F = (2x-12-\lambda(2x), 2y+16-\lambda 2y, -(x^2+y^2-25)) = (0,0,0)$

→ omezená uzavřená množina + spojitá → musí nabývat max a min na množině

$2x(1-\lambda)=12 \quad |^2$

$2y(1-\lambda)=-16 \quad |^2 \ \Sigma$

$x^2+y^2=25$

$\underbrace{(x^2+y^2)}_{25}(1-\lambda)^2=100$

$(1-\lambda)=\pm 2$

$\lambda = \mp 2 - 1$ → $+3$, -1

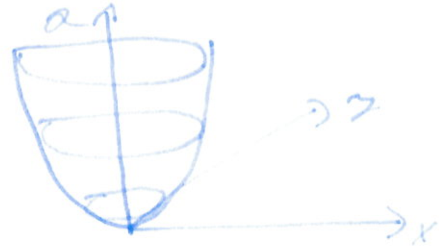
	$\lambda=+3$	$\lambda=-1$
$x = \frac{6}{1-\lambda}$	-3	3
$y = -\frac{8}{1-\lambda}$	4	-4

P_1 $f(-3,4) = 9+16+36+64 = 125$ → MAX

P_2 $f(3,-4) = 9+16-36-64 = -75$ → MIN

vyšlo stejně

(10) $x+y+z$ $\qquad x^2+y^2 \le z \le 1$



→ [illegible]

[illegible]

$\nabla f = (1,1,1)$ → [illegible]

→ $g = x^2+y^2-z=0 \qquad f(x,y,z) = x+y+z$

$\nabla g = (2x, 2y, -1) \quad h=1 \quad \forall x,y,z$ ok

$F = x+y+z - \lambda(x^2+y^2-z)$

$\nabla F = (1-2x\lambda,\ 1-2y\lambda,\ 1+\lambda,\ -(x^2+y^2-z))$

$1 = 2x\lambda \quad \to \quad x=-1/2$

$1 = 2y\lambda \quad \to \quad y=-1/2$

$\lambda = -1 \quad \to \quad z = x^2+y^2 = \frac{1}{2}$, splňuje $z \le 1$

$\underline{f(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}) = -\frac{1}{2}}$

$\underline{z=1}$ (kružnice)

→ $x^2+y^2 \le 1$

$\tilde{f} = x+y+1 \qquad \tilde{g} = x^2+y^2-1=0$

$\nabla\tilde{g}\ (2x, 2y) \to$ pro $(x,y)=(0,0)$ $h=0$

$\underline{f(0,0,1)=1}$

→ pro $(x,y) \ne (0,0)$

$F(x,y,\lambda) = x+y+1-\lambda(x^2+y^2-1)$

$\nabla F = (1-2x\lambda,\ 1-2y\lambda,\ -(x^2+y^2-1))$

$x = \frac{1}{2\lambda} \qquad 2x\lambda = 1 \quad /^2$

$y = \frac{1}{2\lambda} \qquad 2y\lambda = 1 \quad /^2$

$\Sigma \quad 4\lambda^2\underbrace{(x^2+y^2)}_{1} = 2$

$\lambda^2 = \frac{1}{2} \to \lambda = \pm\frac{\sqrt{2}}{2}$

$\Rightarrow (x,y) = \pm\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1\right) = \sqrt{2}+1$

$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1\right) = -\sqrt{2}+1$

→ $x+y-1 \quad x^2+y^2<1$ – [illegible] MIN MAX

→ [illegible]

$f(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}) = -\frac{1}{2}$ MIN $\qquad f(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1) = \sqrt{2}+1$ MAX

(11) kvádr objemu V
→ z kterých $V=abc$ má nejmenší povrch?

$V = xyz$

$S = 2xy + 2yz + 2xz$

$\lim_{x,y,z \to 0} S = \infty$
→ musí nabývat minima

$V = xyz \to z = \frac{V}{xy}$ $\quad x,y \neq 0$

$S = 2\left(xy + \frac{V}{y} + \frac{V}{x}\right)$

$\frac{\partial S}{\partial x} = 2\left(y - \frac{V}{x^2}\right) = 0$ $\qquad y \cdot x^2 = V \implies y x^2 = y^2 x$

$\frac{\partial S}{\partial y} = 2\left(x - \frac{V}{y^2}\right) = 0$ $\qquad y^2 x = V \qquad x = y$

$x = y = z = \sqrt[3]{V}$

Hessián

$$\begin{pmatrix} \frac{4V}{x^3} & 2 \\ 2 & \frac{4V}{y^3} \end{pmatrix}$$

pro $x = y = \sqrt[3]{V}$ $\quad \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$ → pozitivně → definitní → MIN

→ pomocí multiplikátorů

$g = xyz - V = 0$

$\nabla g = (yz, xz, yx)$ pro $(x,y,z) \neq (0,0,0)$ OK $r = 1$

$F = 2(xy + yz + xz) - \lambda(xyz - V)$

$\nabla F = (2(y+z) - \lambda yz,\ 2(x+z) - \lambda xz,\ 2(y+x) - \lambda yx,\ -(xyz - V))$

$\lambda yz = 2(y+z) \quad /x \qquad \lambda V = 2x(y+z)$

$\lambda xz = 2(x+z) \quad /y \qquad \lambda V = 2y(x+z)$

$\lambda yx = 2(x+y) \quad /z \qquad \lambda V = 2z(x+y)$

$2x(y+z) = 2y(x+z) = 2z(x+y)$

$\implies x = y = z$

→ MIN (musí nabývat MIN)

(12) Hranol – n úhelník s maximálním objemem vepsaný do kužele

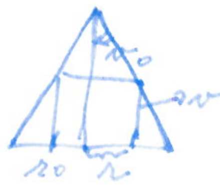
→ n úhelník (pravidelný) $\quad S = \frac{1}{4} n a^2 \cot \frac{\pi}{n}$

$$r_0 = \frac{a}{2 \sin \frac{\pi}{n}}$$

$$S = \frac{1}{4} n \left(2 r \sin \frac{\pi}{n}\right)^2 \frac{\cos \frac{\pi}{n}}{\sin \frac{\pi}{n}} = n \sin \frac{\pi}{n} \cos \frac{\pi}{n} = \frac{n}{2} \sin \frac{2\pi}{n} R_0^2$$

V hranolu

kužel



vztah mezi v a R

$$\operatorname{tg} \alpha = \frac{r_0}{v_0} = \frac{r_0 - R}{v}$$

$$\frac{v}{v_0} = \frac{r_0 - R}{r_0}$$

$$\frac{v}{v_0} = 1 - \frac{R}{r_0}$$

vazba

$$V = \frac{n}{2} \sin \frac{2\pi}{n} R^2 \cdot v$$

$$f(R, v) = \frac{n}{2} \sin \frac{2\pi}{n} R^2 \cdot v$$

$$g = \frac{v}{v_0} + \frac{R}{r_0} - 1 = 0$$

$$v = v_0 \left(1 - \frac{R}{r_0}\right)$$

$$V = \frac{n}{2} \sin \frac{2\pi}{n} R^2 v_0 \left(1 - \frac{R}{r_0}\right) = \frac{n}{2} \sin \frac{2\pi}{n} v_0 \left(R^2 \left(1 - \frac{R}{r_0}\right)\right)$$

$$\frac{\partial V}{\partial R} = \underbrace{\frac{n}{2} \sin \frac{2\pi}{n} v_0}_{A} \left(2R\left(1 - \frac{R}{r_0}\right) + R^2\left(-\frac{1}{r_0}\right)\right) =$$

$$= A \cdot R\left(1 - \frac{R}{r_0} - \frac{R}{r_0}\right) = AR\left(1 - \frac{2R}{r_0}\right) \stackrel{!}{=} 0$$

$V \nearrow \searrow$

$A \neq 0 \quad R \neq 0$

$$R = \frac{r_0}{2}$$

$$\Rightarrow v = \frac{v_0}{2}$$

13) [illegible] (p, q, r) [illegible] $ax+by+cz+d=0$

$$d(x,y,z)=\sqrt{(x-p)^2+(y-q)^2+(z-r)^2}$$

$\sqrt{\ }$ monotoniczna → [illegible] d^2

$$f=(x-p)^2+(y-q)^2+(z-r)^2$$

$$g=ax+by+cz+d=0$$

$\nabla g=(a,b,c)$ [illegible] $a=b=c=0$ [illegible]

$$F(x,y,z,\lambda)=(x-p)^2+(y-q)^2+(z-r)^2-\lambda(ax+by+cz+d)=0$$

$$\nabla F=\big(2(x-p)-\lambda a,\ 2(y-q)-\lambda b,\ 2(z-r)-\lambda c,\ (ax+by+cz+d)\big)$$

$$=(0,0,0,0)$$

$2(x-p)=\lambda a \quad /x \quad /a$

$2(y-q)=\lambda b \quad /y \quad /b$

$2(z-r)=\lambda c \quad /z \quad /c \quad \Sigma$

$$\lambda\underbrace{(ax+by+cz)}_{-d}=2x(x-p)+2y(y-q)+2z(z-r)$$

$$\lambda=-\frac{2}{d}\big(x(x-p)+y(y-q)+z(z-r)\big)$$

$$2(\underbrace{ax+by+cz}_{-d}-pa-qb-rc)=\lambda(a^2+b^2+c^2)$$

$$\lambda=-2\,\frac{(pa+qb+rc+d)}{a^2+b^2+c^2}$$

$$x=\frac{\lambda a}{2}+p=-a\,\frac{pa+qb+rc+d}{a^2+b^2+c^2}+p$$

$$y=\frac{\lambda b}{2}+q=-b\,\frac{pa+qb+rc+d}{a^2+b^2+c^2}+q$$

$$z=-c\,\frac{pa+qb+rc+d}{a^2+b^2+c^2}+r$$

$$l=\sqrt{\left(\frac{pa+qb+rc+d}{a^2+b^2+c^2}\right)^2(a^2+b^2+c^2)}=\frac{|pa+qb+rc+d|}{\sqrt{a^2+b^2+c^2}}$$

(14) vzdálenost mimoběžek

$x = x_1 + at \qquad x = x_2 + ps$

$y = y_1 + bt \qquad y = y_2 + qs$

$z = z_1 + ct \qquad z = z_2 + rs$

$$f = v^2 = (x_1 + at - x_2 - ps)^2 + (y_1 + bt - y_2 - qs)^2 + (z_1 + ct - z_2 - rs)^2 \qquad f(s, t)$$

$$\frac{\partial f}{\partial t} = 2a\underbrace{(x_1 + at - x_2 - ps)}_{V_x} + 2b\underbrace{(y_1 + bt - y_2 - qs)}_{V_y} + 2c\underbrace{(z_1 + ct - z_2 - rs)}_{V_z} = 0$$

$$\frac{\partial f}{\partial s} = -2p(x_1 + at - x_2 - ps) - 2q(V_y) - 2r(V_z) = 0$$

$$2aV_x + 2bV_y + 2cV_z = 0$$

$$-2pV_x - 2qV_y - 2rV_z = 0$$

$$l = \sqrt{v^2} = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

MATICOVĚ

$$\underbrace{\begin{pmatrix} 2(a^2+b^2+c^2) & -2(ap+bq+cr) \\ -2(pa+qb+rc) & -2(p^2+q^2+r^2) \end{pmatrix}}_{A} \begin{pmatrix} t \\ s \end{pmatrix} + \underbrace{\begin{pmatrix} 2(a(x_1-x_2) + b(y_1-y_2) + c(z_1-z_2)) \\ -2(p(x_1-x_2) + q(y_1-y_2) + r(z_1-z_2)) \end{pmatrix}}_{-b} = 0$$

$$\begin{pmatrix} t \\ s \end{pmatrix} = A^{-1} \cdot b$$

$$\begin{pmatrix} t \\ s \end{pmatrix} = \frac{1}{|A|} \begin{pmatrix} -2(p^2+q^2+r^2) & 2(ap+bq+cr) \\ 2(ap+bq+cr) & 2(a^2+b^2+c^2) \end{pmatrix} b$$

→ pak dosazení do $d = \sqrt{v^2} = |v|$

15. AG nerovnost — pomocí vázaných extrémů + minimum

$$\frac{a_1+a_2+\dots+a_n}{n} \geq \sqrt[n]{a_1\cdots a_n} \qquad a_i \geq 0$$

BÚNO (vydělení konstantou)

$a_1 \cdot a_2 \cdots a_n = 1$

$$f(a_1, a_2, \dots a_n) = \frac{a_1+a_2+\dots+a_n}{n}$$

$g = a_1 a_2 \dots a_n - 1 = 0$

$\nabla g = (a_2 \dots a_n, a_1 a_3 \dots a_n, \dots, a_1 \dots a_{n-1})$ nulový nebude, pro $a_i = 0$ (bude to nula)

$F = f - \lambda g$

$\nabla F = \left(\frac{1}{n} - \lambda a_2 \dots a_n, \frac{1}{n} - \lambda a_1 a_3 \dots a_n, \dots, \frac{1}{n} - \lambda a_2 \dots a_n, -g\right) = 0$

$\frac{1}{n} = \lambda a_2 \dots a_n \quad / a_1$

$\frac{1}{n} = \lambda a_1 a_3 \dots a_n \quad / a_2$

$\vdots$

$\frac{1}{n} = \lambda a_2 \dots a_n \quad / a_n$

$\lambda = \frac{a_1}{n} = \frac{a_2}{n} = \frac{a_3}{n} \dots$

$\Rightarrow a_1 = a_2 = a_3 = \dots a_n = n\lambda$

$\rightarrow$ musí to nastávat min nebo max

$f(a_1, a_2, \dots, a_n) = \lambda n$

$a_1 \cdot \dots a_n = 1$

$\sqrt[n]{n\lambda \cdot n\lambda \dots n\lambda} = 1$

$n\lambda = 1 \rightarrow \lambda = \frac{1}{n}$

uvažujme nasř $(2, \frac{1}{2}, 1 \dots 1)$

$f(2, \frac{1}{2}, 1, \dots, 1) = \frac{2+\frac{1}{2}+n-2}{n} = 1 + \frac{1}{2n} > 1 \rightarrow$ bude

tedy je to minimum $\boxed{\checkmark}$

Hölderova nerovnost

⊛ $\sum x_i y_i \le \left(\sum x_i^p\right)^{1/p} \left(\sum y_i^q\right)^{1/q}$ $\quad x_i, y_i \ge 0$

$p > 1$ a $\frac{1}{p} + \frac{1}{q} = 1$

→ BÚNO $\tilde{x}_i = \dfrac{x_i}{\left(\sum x_j^p\right)^{1/p}}$ $\quad \tilde{y}_i = \dfrac{y_i}{\left(\sum y_j^q\right)^{1/q}}$

→ ⊛ → $\sum \tilde{x}_i \tilde{y}_i \le 1$ z předpokladu

pro 2 vazby $\quad g_1 = \sum \tilde{x}_i^p - 1 = 0$

$\quad g_2 = \sum \tilde{y}_i^q - 1 = 0$

$\nabla g_1 = \left(p \tilde{x}_1^{p-1}, \dots, p \tilde{x}_n^{p-1}, 0, \dots, 0\right)$

$\nabla g_1 = \left(0, \dots, 0, q \tilde{y}_1^{q-1}, \dots, q \tilde{y}_n^{q-1}\right)$

hodnost 2 pro nenulové $\tilde{x}_i, \tilde{y}_i$, pro nulové triviální platnost

$F = f - \lambda_1 g_1 - \lambda_2 g_2$

$\dfrac{\partial F}{\partial x_j} = \tilde{y}_j - \lambda_1 p \tilde{x}_j^{p-1} = 0 \quad /^q + \sum$

$\dfrac{\partial F}{\partial y_j} = \tilde{x}_j - \lambda_2 q \tilde{y}_j^{q-1} = 0 \quad /^p + \sum$

+2 vazby

$\sum \tilde{y}_j^q = \lambda_1^q p^q \sum \tilde{x}_j^{(p-1)q} = 1 = \lambda_1^q p^q \underbrace{\sum \tilde{x}_j^p}_{1} \Rightarrow \lambda_1 = \frac{1}{p}$

$\sum \tilde{x}_j^p = \lambda_2^p q^p \sum \tilde{y}_j^{(q-1)p} = 1 = \lambda_2^p q^p \underbrace{\sum \tilde{y}_j^q}_{1} \Rightarrow \lambda_2^p q^p = 1$

$\lambda_2 = \frac{1}{q}$

a $\frac{1}{p} + \frac{1}{q} = 1 \rightarrow q = \dfrac{p}{p-1}$

$p = \dfrac{q}{q-1}$

$\tilde{y}_j = \overset{\frac{1}{p}}{\lambda_1} p \tilde{x}_j^{p-1} = \boxed{\begin{matrix}\tilde{x}_j^{p-1} = \tilde{y}_j \\ \tilde{y}_j^{q-1} = \tilde{x}_j\end{matrix}}$ body vyšetříme zvlášť

$\tilde{x}_j = \underset{\frac{1}{q}}{\lambda_2} q \tilde{y}_j^{q-1} =$

kritické splňují krajní hodnoty, ale musíme ukázat, že je maximum

$\sum \tilde{x}_i \tilde{y}_i = \sum \tilde{x}_i \tilde{x}_i^{p-1} = \sum \tilde{x}_i^p = 1$

$\dfrac{\partial^2 f}{\partial x_j^2} = \dfrac{\partial g}{\partial x_j} = \dfrac{\partial}{\partial x_j} x_j^{p-1} = (p-1) x_j^{p-2}$

$\dfrac{\partial^2 f}{\partial x_i \partial x_j} = 0$

$\dfrac{\partial^2 f}{\partial x_j \partial y_j} = 1$

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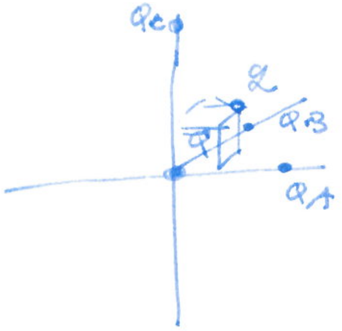

$P_A = 3, 0, 0$

$P_B = 0, 3, 0$

$P_C = 0, 0, 4$

$P_Q = 1, 1, 1$

→ v rovnovážní poloze musí pot. energie nabývat ~~min~~ lokálních extrémů, konkrétně buď lokálního minima

$$V = \frac{1}{4\pi\varepsilon_0} \frac{Q_\alpha}{|\vec{R} - \vec{R}_\alpha|}$$

$$\tilde{V} = qV$$

$$V = V_A + V_B + V_C + V_Q$$

$$= \frac{1}{4\pi\varepsilon_0} \left(\frac{Q}{\sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}} + \frac{Q_A}{\sqrt{(x-3)^2 + y^2 + z^2}} + \frac{Q_B}{\sqrt{x^2 + (y-3)^2 + z^2}} + \frac{Q_C}{\sqrt{x^2 + y^2 + (z-4)^2}} \right)$$

$$\nabla V = -\left(\frac{Q}{R_Q^3} x + \frac{Q_A (x-3)}{R_A^3} + \frac{Q_B (x)}{R_B^3} + \frac{Q_C x}{R_C^3} \right)$$

⋮

chceme, aby toto nulové v (1, 1, 1)

$R_Q (1,1,1) = \sqrt{3}$

$R_a (1,1,1) = \sqrt{6} = \sqrt{3}\sqrt{2}$

$R_b (1,1,1) = \sqrt{6}$

$R_c (1,1,1) = \sqrt{11}$

$$\frac{Q}{(\sqrt{3})^3} + Q_A \frac{(-2)}{(\sqrt{6})^3} + \frac{Q_B}{(\sqrt{6})^3} + \frac{Q_C}{(\sqrt{11})^3} = 0$$

$$\frac{Q}{\sqrt{3}^3} + \frac{Q_A}{(\sqrt{6})^3} + \frac{Q_B(-2)}{(\sqrt{6})^3} + \frac{Q_C}{(\sqrt{11})^3} = 0$$

$$\frac{Q}{\sqrt{3}^3} + \frac{Q_A}{(\sqrt{6})^3} + \frac{Q_B}{(\sqrt{6})^3} - \frac{3 Q_C}{\sqrt{11}^3} = 0$$

$$\sum_{1-3} \frac{Q}{\sqrt{3}^3} = \frac{2}{(\sqrt{11})^3} Q_C$$

$$Q_C = Q \left(\sqrt{\frac{11}{3}}\right)^3$$

$$-\frac{3 Q_A}{(\sqrt{6})^3} = -\frac{3 Q_B}{(\sqrt{6})^3} \Rightarrow Q_A = Q_B$$

$$2 \frac{Q}{(\sqrt{3})^3} - \frac{Q_A}{(\sqrt{6})^3} = 0$$

$Q_A = Q_B = 4\sqrt{2}\, Q$ — v–k jinou

(17) $x = u(v,w)$ $\quad \frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} = 0$ $\quad \frac{\partial v}{\partial y} = 0$

$y = v$ $\quad w_x = z_x$

$z(x,y) = w$ $\quad w_y = z_y$

$\boxed{u, v, w}$ + funkce zal. nezávislých proměnných

$v = \tilde{v}(x,y)$

$w = w(x,y)$

$u = \tilde{u}(v(x,y), w(x,y))$

$\frac{\partial}{\partial x}$ $\quad x = u(v,w)$

$1 = u_v v_x + u_w w_x = \boxed{u_w z_x = 1}$

$\frac{\partial}{\partial y}$ $\quad x = u(v,w)$

$0 = u_v v_y + u_w w_y = 0 = \boxed{u_v + u_w z_y = 0}$

$\frac{\partial}{\partial x}$ $\quad u_w z_x = 1$

$u_{vw} \underbrace{v_x}_{0} z_x + u_{ww}(z_x)^2 + u_w z_{xx} = 0$

$\boxed{u_{ww}(z_x)^2 + u_w z_{xx} = 0}$

$\frac{\partial}{\partial y}$ $\quad u_w z_x = 1$

$u_{vw} \underbrace{v_y}_{1} z_x + u_{ww} z_x z_y + u_w z_{xy} = 0$

$\boxed{u_{vw} z_x + u_{ww} z_x z_y + u_w z_{xy} = 0}$

$\frac{\partial}{\partial y}$ $\quad u_v + u_w z_y = 0$

$u_{vv} \underbrace{v_y}_{1} + u_{vw} \underbrace{w_y}_{z_y} + u_{wv} \underbrace{v_y}_{1} z_y + u_{ww}(z_y)^2 + u_w z_{yy} = 0$

$\boxed{u_{vv} + 2u_{wv} z_y + u_{ww}(z_y)^2 + u_w z_{yy} = 0}$

$z_y^2 z_{xx} - 2 z_x z_y z_{xy} + (z_x)^2 z_{yy} = 0$

$-(z_y)^2 \left(\frac{u_{ww}(z_x)^2}{u_w} \right) + 2 z_x z_y \frac{u_{vw} z_x + u_{ww} z_x z_y}{u_w} +$

$+ (z_x)^2 \frac{u_{vv} + 2u_{wv} z_y + u_{ww}(z_y)^2}{u_w} = 0$

$-(z_x)^2 \frac{u_{vv}}{u_w} = 0 \quad \longrightarrow \quad -\frac{u_{vv}}{(u_w)^3} = 0 \quad u_w \neq 0$

$u_{vv} = 0 \Rightarrow X = u = y\,\varphi(z) + \psi(z)$

(18) f_x $\quad \nabla f = (f_x, f_y, f_z)$ w sferycznych współrzędnych

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$f(r,\theta,\varphi) = \tilde{f}(r(x,y,z), \theta(x,y,z), \varphi(x,y,z))$$

(1) $f_r = f_x \sin\theta\cos\varphi + f_y \sin\theta\sin\varphi + f_z\cos\theta$ $\quad | \, r\sin\theta$

(2) $f_\theta = f_x r\cos\theta\cos\varphi + f_y r\cos\theta\sin\varphi - f_z r\sin\theta$ $\quad | \cos\theta$

(3) $f_\varphi = -f_x r\sin\theta\sin\varphi + f_y r\sin\theta\cos\varphi + 0$

$r\sin\theta$ (1) $+ \cos\theta$ (2)

$$r\sin\theta f_r + \cos\varphi f_\theta = f_x r\sin^2\theta\cos\varphi + f_y r\sin^2\theta\sin\varphi + f_x r\cos^2\theta\cos\varphi + f_y r\cos^2\theta\sin\varphi = f_x r\cos\varphi + f_y r\sin\varphi \quad (4)$$

$-\sin\theta\cos\varphi$ (4) $+$ (3) $\sin\varphi$

$$-r\sin^2\theta\cos\varphi f_r - \cos^2\varphi\sin\theta f_\theta + \sin\varphi f_\varphi = -f_x r\cos^2\varphi\sin\theta - f_x r\sin^2\varphi\sin\theta = -f_x r\sin\theta$$

$$\Rightarrow f_x = \sin\theta\cos\varphi f_r + \frac{\cos^2\varphi\sin\theta}{r} f_\theta - \frac{\sin\varphi}{\sin\theta \cdot r} f_\varphi$$

19 $x^2 z_x + y^2 z_y = z^2$

$u = x$

$v = \frac{1}{y} - \frac{1}{x}$

$\underbrace{w = \frac{1}{z} - \frac{1}{x}}$

det
$$\begin{vmatrix} 1 & 0 & 0 \\ +\frac{1}{x^2} & -\frac{1}{y^2} & 0 \\ \frac{1}{x^2} & 0 & -\frac{1}{z^2} \end{vmatrix} = \frac{1}{y^2 z^2} \neq 0$$

$\to$ inverse

$$\begin{matrix} 1 & 0 & 0 \\ \frac{1}{x^2} & -\frac{1}{y^2} & 0 \end{matrix}$$

$J = x^2 y^2 \neq 0$

hledáme $w(u,v)$

$u = x$ $\quad \frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 0 \quad u(x,y)$

$v = \frac{1}{y} - \frac{1}{x} \quad \frac{\partial v}{\partial x} = \frac{1}{x^2} \quad \frac{\partial v}{\partial y} = -\frac{1}{y^2} \quad v(x,y)$

$w(u,v) = \frac{1}{z(x,y)} - \frac{1}{x} \quad / \frac{\partial}{\partial x}$

$\frac{\partial}{\partial x}$: $w_u u_x + w_v v_x = -\frac{1}{z^2} z_x + \frac{1}{x^2}$ $\to$ (1) $w_u + \frac{1}{x^2} w_v = -\frac{1}{z^2} z_x + \frac{1}{x^2}$

$\frac{\partial}{\partial y}$: $w_u u_y + w_v v_y = -\frac{1}{z^2} z_y$ $\to$ (2) $0 - \frac{1}{y^2} w_v = -\frac{1}{z^2} z_y$

(1) $\cdot\, x^2 z^2 +$ (2) $z^2 y^2$

$z^2 x^2 w_u + z^2 w_v = -x^2 z_x + z^2$

$+ \qquad -z^2 w_v = -y^2 z_y$

$z^2 x^2 w_u = \underbrace{-x^2 z_x - y^2 z_y + z^2}_{0 \text{ (zadání)}}$

pro $z \neq 0, x \neq 0$ $w_u = 0$

(20) $z_{xx} - z_{yy} = 0 \qquad u = \frac{x}{x^2+y^2} \qquad v = -\frac{y}{x^2+y^2} \quad | \; z = w$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad ; \quad \frac{\partial^2 u}{\partial x^2} = \cdot$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = +\frac{2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = -\frac{x^2+y^2-y(2y)}{(x^2+y^2)^2} = -\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\begin{vmatrix} \frac{y^2-x^2}{x^2+y^2} & -\frac{2xy}{(x^2+y^2)^2} \\ \frac{2xy}{(x^2+y^2)^2} & +\frac{y^2-x^2}{(x^2+y^2)^2} \end{vmatrix} \neq 0 \qquad \begin{aligned} &+(y^2-x^2)^2 + 4x^2y^2 \\ &+(x^4 - 2x^2y^2+y^4) + 4x^2y^2 \neq 0 \\ &\quad x^4+y^4 \neq -2x^2y^2 \end{aligned}$$

$$\frac{\partial z}{\partial x} = \frac{\partial w}{\partial u} u_x + w_v v_x = w_u \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} w_v$$

$$z_y = w_u u_y + w_v v_y = w_u \frac{-2xy}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2} w_v$$

$$\begin{aligned} z_{xx} = (w_u u_x + w_v v_x)_{,x} &= \underline{w_{uu} (u_x)^2} + w_u u_{xx} + \\ &+ w_{uv} u_x v_x \\ &+ w_{vv} (v_x)^2 + w_{vu} v_x u_x + \\ &+ w_v v_{xx} = \\ &= \underline{w_{uu} (u_x)^2} + \underline{w_{vv} (v_x)^2} \\ &+ 2 w_{uv} (u_x)(v_x) + w_v v_{xx} + \\ &+ w_u u_{xx} \end{aligned}$$

$$\begin{aligned} z_{yy} &= \underline{w_{uu} (u_y)^2} + \underline{w_{vv} (v_y)^2} \\ &+ 2 w_{uv} (u_y)(v_y) + w_u u_{yy} + \\ &+ w_v v_{yy}. \end{aligned}$$

$$\Sigma \qquad u_x = v_y \qquad u_y = -v_x$$

$z_{xx} + z_{yy} =$

$$\begin{aligned} &= w_{uu} (u_x^2 + u_y^2) + w_{vv} (v_x^2 + v_y^2) + w_u (u_{xx} + u_{yy}) + \\ &+ w_v (v_{xx} + v_{yy}) \\ &= w_{uu} (u_x^2 + u_y^2) + w_{vv} (u_y^2 + u_x^2) + w_u \overbrace{(u_{xx} + u_{yy})}^{=0} + \\ &\qquad\qquad\qquad\qquad\qquad\qquad v_{yx} \;\; -v_{xy} \\ &+ w_v \underbrace{(-u_{yx} + u_{xy})}_{0} = (w_{uu} + w_{vv})(u_x^2 + u_y^2) = 0 \end{aligned}$$

as $u_x^2 + u_y^2 \neq 0$

$$\boxed{w_{uu} + w_{vv} = 0}$$

(2.1) $x^2 z_{xx} - (x^2+y^2) z_{xy} + y^2 z_{yy} = 0$

$u = x+y$

$v = \frac{1}{x} + \frac{1}{y}$

$z(x,y) = z(x(u,v), y(u,v)) = w(u,v)$ $\qquad z = w$

$$\begin{vmatrix} 1 & 1 & 0 \\ -\frac{1}{x^2} & -\frac{1}{y^2} & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0 \qquad -\frac{1}{y^2} + \frac{1}{x^2} \neq 0$$

$x^2 \neq y^2$

$$\begin{matrix} 1 & 1 & 0 \\ -\frac{1}{x^2} & -\frac{1}{y^2} & 0 \end{matrix}$$

$\frac{\partial w}{\partial x} = w_u u_x + w_v v_x$

$z_y = w_u u_y + w_v v_y$

$z_{xx} = w_{uu}(u_x)^2 + w_{uv} u_x v_x + w_u u_{xx}$

$w_{uv} u_x v_x + w_{vv}(v_x)^2 + w_v v_{xx}$

$= w_{uu}(u_x^2) + 2 w_{uv} u_x v_x + w_{vv}(v_x)^2$

$+ w_u u_{xx} + w_v v_{xx}$

$z_{xy} = w_{uu} u_x u_y + w_{uv} u_x v_y + w_u u_{xy}$

$+ w_{vu} v_x u_y + w_{vv} v_x v_y + w_v v_{xy}$

$z_{yy} = w_{uu}(u_y^2) + 2 w_{uv} v_y u_y + w_{vv}(v_y)^2 +$

$+ w_u u_{yy} + w_v v_{yy}$

$\frac{\partial u}{\partial y} = 1$

$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial x \partial y} = 0$

$\frac{\partial v}{\partial x} = -\frac{1}{x^2} \qquad \frac{\partial^2 v}{\partial x \partial y} = 0 \qquad \frac{\partial^2 v}{\partial x^2} = \frac{2}{x^3}$

$\frac{\partial^2 v}{\partial y^2} = \frac{2}{y^3}$

$\frac{\partial v}{\partial y} = -\frac{1}{y^2}$

$z_{xx} = w_{uu} + 2 w_{uv} \cdot 1 \cdot \left(-\frac{1}{x^2}\right) + w_{vv}\left(-\frac{1}{x^2}\right)^2 + 0 + w_v\left(\frac{2}{x^3}\right)$

$z_{xy} = w_{uu} \cdot 1 \cdot 1 + w_{uv} \cdot 1 \cdot \left(-\frac{1}{y^2}\right) + w_u \cdot 0 + w_{vu}\left(-\frac{1}{x^2}\right) \cdot 1$

$+ w_{vv}\left(-\frac{1}{x^2} \cdot \frac{1}{y^2}\right) + w_v \cdot 0$

$z_{yy} = w_{uu} + 2 w_{uv}\left(-\frac{1}{y^2}\right) \cdot 1 + w_{vv}\left(-\frac{1}{y^2}\right)^2 + w_u \cdot 0 + w_v \frac{2}{y^3}$

$x^2 z_{xx} - (x^2+y^2) z_{xy} + y^2 z_{yy} = 0$

$(x^2+y^2) w_{uu} - 4 w_{uv} + w_{vv}\left(+\frac{1}{x^2} + \frac{1}{y^2}\right) + 2 w_v \Big(\overbrace{\frac{1}{x} + \frac{1}{y}}^{v}\Big) +$

$- (x^2+y^2) w_{uu} - (x^2+y^2)\left(-\frac{1}{y^2}\right) w_{uv} - w_{uv}\left(-\frac{1}{x^2}\right)(x^2+y^2) - w_{vv} \frac{x^2+y^2}{-x^2 y^2}$

$w_{uv}\left((x^2+y^2)\left(\frac{1}{x^2} + \frac{1}{y^2}\right) - 4\right) + 2 w_v v = 0$

$$w_{uv} = + \frac{2v w_v}{4 - (x^2+y^2)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)} \quad \frac{2v w u}{4uv - u^2v^2} = \frac{2 w u}{u(4-uv)}$$

$$uv = (x+y)\left(\frac{1}{x}+\frac{1}{y}\right) = 1 + \frac{x}{y} + \frac{y}{x} + 1 = 2 + \frac{x^2+y^2}{xy}$$

$$4uv - u^2v^2 = 4\left(2 + \frac{x^2+y^2}{xy}\right) - 4 - 4\,\frac{x^2+y^2}{xy} + \frac{(x^2+y^2)^2}{x^2y^2} =$$

$$= 4 - \frac{(x^2+y^2)^2}{x^2y^2} = 4 - (x^2+y^2)\left(\frac{1}{x^2}+y^2\right)$$

$$(x^2+y^2)\left(\frac{1}{x^2}+\frac{1}{y^2}\right) = \frac{(x^2+y^2)^2}{x^2y^2}$$