L9/1

Př. 1

STANDARD

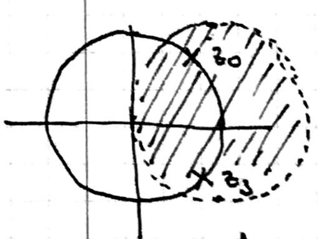
$$\int_{\partial\sigma} \frac{dz}{z^4+1} \quad \text{kde } \sigma = \{ z = x+iy \,;\, x^2+y^2 \le 2x \}$$

$$\Updownarrow$$

$$(x-1)^2 + y^2 \le 1$$

$$z^4+1 = 0 \iff e^{4i\alpha} = -1 \qquad 4\alpha = \pi + 2k\pi \qquad k = 0,1,2,3$$

$$\alpha_k = \frac{\pi}{4} + k\frac{\pi}{2}$$



$$\alpha_k = \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\}$$

$$= \left\{ \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} \right\}$$

$$\int_{\partial\sigma} \frac{dz}{z^4+1} = 2\pi i \left\{ \operatorname{Res}_{z_0} f(z) + \operatorname{Res}_{z_3} f(z) \right\}$$

$$= 2\pi i \left\{ \frac{1}{4z_0^3} + \frac{1}{4z_3^3} \right\} = \frac{\pi i}{2} \left\{ -z_0 - z_3 \right\}$$

$$= -\frac{\pi i}{2} \left\{ \frac{\sqrt{2}}{2} + \cancel{i\frac{\sqrt{2}}{2}} + \frac{\sqrt{2}}{2} - \cancel{i\frac{\sqrt{2}}{2}} \right\}$$

$$= -\frac{\sqrt{2}\pi}{2} i$$

INTEGRÁLY FUNKCÍ KLESAJÍCÍ V $\infty$ RYCHLEJI NEŽ $\frac{1}{R}$

2

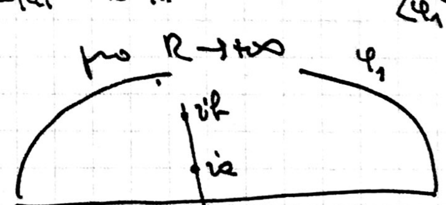
a) $$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)} \qquad a, b > 0 \qquad a \neq b \qquad I = \int_{-\infty}^{+\infty} f(z)\,dz$$

$$|z^2+a^2| \ge |z|^2 - |a|^2 \quad \& \quad |z| = R$$

$$|f(z)| \le \frac{1}{(|z|^2-|a|^2)} \frac{1}{(|z|^2-|b|^2)} = \frac{1}{R^2-|a|^2} \frac{1}{R^2-|b|^2} \overset{\text{JORDAN}}{\Longrightarrow} \int_{\langle\varphi_1\rangle} f(z) \to 0 \quad R \to \infty$$

$$|f(z)|\,R \le \frac{R}{(R^2-|a|^2)(R^2-|b|^2)} \to 0 \quad \text{pro } R \to +\infty$$

$a \neq b$



$$\int_{-\infty}^{+\infty} f(z)\,dz = 2\pi i \left( \operatorname{Res}_{ia} f + \operatorname{Res}_{ib} f \right)$$

$\varphi_0(t) = t \quad t \in \langle -R, R \rangle$

$\varphi_1(t) = Re^{it} \quad t \in \langle 0, \pi \rangle$

$$= 2\pi i \left( \frac{1}{-a^2+b^2} \frac{1}{2ia} + \frac{1}{2ib} \frac{1}{a^2-b^2} \right)$$

$$= \frac{\pi}{b^2-a^2} \left( \frac{1}{a} - \frac{1}{b} \right) = \frac{\pi}{ab} \frac{1}{(b+a)} \xrightarrow{b \to a} \frac{\pi}{2a^3} \quad \text{shoduje s výsledkem níže}$$

b) $$f(z) = \frac{1}{(z^2+a^2)^2} \qquad \text{v } ia \text{ pól násobnosti } 2$$

$\boxed{a = b}$

$$\operatorname{Res}_{ia} f = \frac{1}{1!} \lim_{z \to ia} \left[ \frac{1}{(z^2+a^2)^2} (z-ia)^2 \right]'$$

$$= \lim_{z \to ia} \left[ \frac{1}{(z+ia)^2} \right]' = \lim_{z \to ia} \frac{-2}{(z+ia)^3}$$

$$\int_{\partial\sigma} f(z)\,dz = \frac{\pi}{2a^3}$$

$$= -\frac{2}{8(i)^3 a^3} = \frac{1}{4a^3 i}$$



OTOČ %

Lze tedy řešit také tak, že prvedu

$$\lim_{b\to a} I(a,b) = \lim_{b\to a} \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} \underset{\text{NEB}}{\overset{=}{\uparrow}} \int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)^2}$$

$x^2+b^2 \geq |x|^2 \quad \wedge \quad |b-a| < \frac{1}{2} \Leftrightarrow a-\frac{1}{2} < b < a+\frac{1}{2}$

$b^2 \leq a^2 + \frac{1}{4} + a \leq \frac{3}{2}a^2 + \frac{3}{4}$

$\geq |x|^2 - \frac{3}{2}a^2 - \frac{3}{4}$

$b \geq a-\varepsilon \quad b^2 \geq a^2 - 2\varepsilon a + \varepsilon^2$

$\int_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2}$ $\quad x^2+b^2 \geq \boxed{x^2+\frac{a^2}{2}+\frac{\varepsilon^2}{2}}$

$\varepsilon a \leq \frac{1}{2}\varepsilon^2 + \frac{1}{2}a^2$

$\geq \frac{1}{2}a^2 + \frac{1}{2}\varepsilon^2$

$$\frac{1}{(x^2+a^2)} \frac{1}{(x^2+\frac{a^2}{2}+\frac{\varepsilon^2}{2})} \overset{\varepsilon=a}{=} \frac{1}{(x^2+a^2)^2} \in \mathcal{L}^1(-\infty,\infty)$$

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$\blacksquare$

c) $I = \int_0^\infty \frac{x^{m-1}}{x^n+1}\,dx$ kde $\boxed{n > m}$

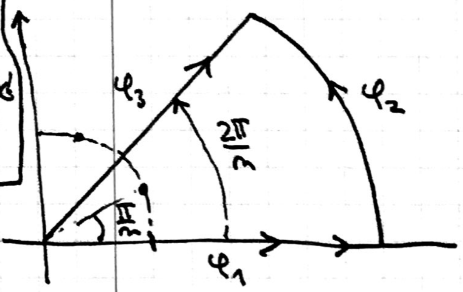
$f(z) = \frac{z^{m-1}}{z^n+1}$ má n jednoduchých kořenů

$z_0, \dots, z_{n-1} = e^{i\varphi_0}, \dots, e^{i\varphi_{n-1}}$

kde $\varphi_k = \frac{\pi}{n} + \frac{2k\pi}{n}$ $\quad k = 0, 1, \dots, n-1$

$= \frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}, \dots, 2\pi - \frac{\pi}{n}$

PROČ TATO OBLAST BUDE PATRNÉ ZA CHVÍLI




$\langle\varphi_1\rangle$: $t$ $\quad t \in \langle 0, R\rangle$

$\langle\varphi_2\rangle$: $Re^{it}$, $t \in \langle 0, \frac{2\pi}{n}\rangle$

$\langle\varphi_3\rangle$: $te^{i\frac{2\pi}{n}}$ $\quad t \in \langle 0, R\rangle$

$$\int_{\langle\varphi_1\rangle} + \int_{\langle\varphi_2\rangle} - \int_{\langle\varphi_3\rangle} = \int_{\partial\Omega} f(z) = 2\pi i \operatorname{Res}_{e^{i\frac{\pi}{n}}} f(z)$$

R: $\int_{\langle\varphi_1\rangle} \to I$, $\int_{\langle\varphi_2\rangle} \xrightarrow{\text{JORDAN}} 0$, $\int_{\langle\varphi_3\rangle} \to ?$

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$$-\frac{2\pi i}{n}\,\frac{\left(e^{i\frac{\pi}{n}}\right)^{m-1} e^{i\frac{\pi}{n}}}{} = -\frac{2\pi}{n} e^{i\frac{\pi m}{n}} i$$

$$\int_{\langle\varphi_3\rangle} = \int_0^R \frac{t^{m-1} e^{i\frac{2\pi}{n}m} e^{i\frac{2\pi}{n}}}{t^n \underbrace{e^{i2\pi}}_{=1} + 1}\,dt = e^{i\frac{2\pi}{n}m} \int_0^R \frac{t^{m-1}}{t^n+1}\,dt \xrightarrow[R\to\infty]{} e^{i\frac{2\pi}{n}m} I$$

Tedy:

$$\left(1 - e^{i\frac{2\pi m}{n}}\right) I = -\frac{2\pi}{n} i\, e^{i\frac{\pi m}{n}}$$

$\Updownarrow$

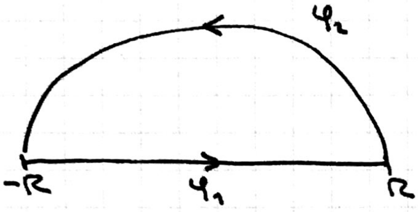
$$I = \frac{\pi}{n} \frac{1}{\frac{e^{i\frac{\pi m}{n}} - e^{-i\frac{\pi m}{n}}}{2i}} = \frac{\pi}{n} \frac{1}{\sinh\left(i\frac{\pi m}{n}\right)} = \frac{\pi}{n} \frac{1}{\sin\frac{\pi m}{n}}$$

③ INTEGRÁLY TYPU $f(z)e^{i\gamma z}$, $\gamma > 0$ a $f(z) \sim \frac{1}{z^s}$ v $\infty$, $s \in (0,1)$

a) $I = \int_0^\infty \frac{x \sin ax}{x^2+b^2}\,dx \overset{\text{sudá fce}}{=} \frac{1}{2}\int_{-\infty}^{+\infty} \frac{x \sin ax}{x^2+b^2} = \frac{1}{2}\mathcal{I}.$

NELZE TENTO POSTUP UŽÍT PRO cos ax !

Ad $\mathcal{I}$ $\quad \sigma$



$$f(z) = \frac{z e^{iaz}}{z^2+b^2}$$

Residuová věta:

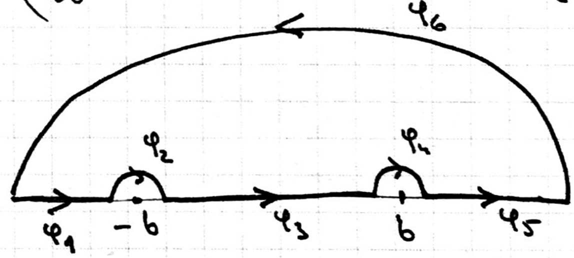
$$\int_{(\varphi_1)} + \int_{(\varphi_2)} = 2\pi i \,\mathrm{Res}_{ib} f(z) = 2\pi i \frac{\cancel{ib}\, e^{-ab}}{2\cancel{ib}} = \pi e^{-ab} i$$

$\downarrow R\to\infty$: $\int_{-\infty}^{+\infty} \frac{x e^{iax}}{x^2+b^2}$ $\qquad$ $\downarrow 0$: Jordanovo lemma, 2. část. ověřit předp.

$$\underline{\underline{I = \frac{\pi}{2} e^{-ab}}}$$

b) $I = \int_{-\infty}^{\infty} \frac{x \sin ax}{x^2-b^2}\,dx \overset{\text{def.}}{=} \lim_{\varepsilon\to 0+}\left(\int_{-\infty}^{-b-\varepsilon} \dots + \int_{-b+\varepsilon}^{b-\varepsilon} \dots + \int_{b+\varepsilon}^{\infty}\right)$ [Integrál ve smyslu hlavní hodnoty

$f(z) = \frac{z e^{iaz}}{z^2-b^2} \in H(\sigma)$ $\quad \sigma =:$



$$\int_{\partial\sigma} f(z)\,dz \overset{\text{dle Cauchyho věty}}{=} 0$$

$$\underbrace{\int_{\varphi_1} + \int_{\varphi_3} + \int_{\varphi_5}}_{\downarrow\, \varepsilon\to0,\ R\to+\infty} - \underbrace{\int_{(\varphi_2)} - \int_{(\varphi_4)}}_{\downarrow\, \varepsilon\to 0,\ \text{Lemma B}} + \int_{(\varphi_6)} = 0$$

$\int_{(\varphi_6)} \to 0$ dle Jordanova lemma 2. část

$$iI \qquad -\left[i\pi \,\mathrm{res}_{-b} f + i\pi\,\mathrm{res}_b f\right] \qquad \Rightarrow \quad I = i\pi\left(\mathrm{res}_b f + \mathrm{res}_{-b} f\right)$$

$$= i\pi\left[\frac{\cancel{b} e^{iab}}{2\cancel{b}} + \frac{\cancel{b} e^{-iab}}{2\cancel{b}}\right] = i\frac{\pi}{\cancel{2}}(\cos ab)$$

$$\underline{I = \pi \cos ab}$$

Takže $\boxed{\int_0^\infty \frac{x \sin ax}{x^2-b^2}\,dx = \frac{\pi}{2}\cos ab}$

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④ $\boxed{\text{Integrály typu } \int_0^{2\pi} f(\cos t, \sin t)\,dt = \int_{|z|=1} F(z)}$

$z=e^{it} \Rightarrow \cos t = \frac{e^{it}+e^{-it}}{2} = \frac{z+\frac{1}{z}}{2}$, $\sin t = \frac{z-\frac{1}{z}}{2i}$

Př: $I(a) = \int_0^{2\pi}\frac{dt}{1-2a\cos t+a^2} = \int_{|z|=1}\frac{dz}{1-a(z+\frac{1}{z})+a^2}\,\frac{1}{iz} =$

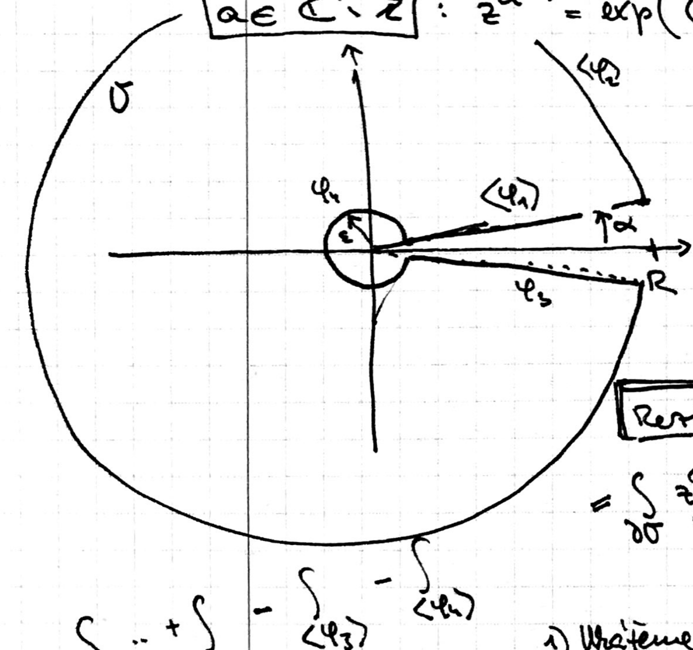
$= i\int_{|z|=1}\frac{dz}{az^2-(a^2+1)z+a} = \int_{|z|=1}\frac{i\,dz}{(az-1)(z-a)}$

1) $|a|<1$ $\quad I(a) = 2\pi i\,\mathrm{res}_a f = \frac{2\pi}{1-a^2}$

2) $|a|>1$ $\quad I(a) = 2\pi i\,\mathrm{res}_{\frac{1}{a}} f = -2\pi \lim_{z\to\frac{1}{a}}\frac{1}{(z-a)a} = \frac{-2\pi}{1-a^2} = \frac{2\pi}{a^2-1}$

⑤ Integrály typu $I(a) = \int_0^\infty z^{a-1}f(z)\,dz$ $\quad$ Mellinova transformace

$\boxed{a\in\mathbb{C}\setminus\mathbb{Z}}$: $z^{a-1} = \exp((a-1)\ln_0 z)$ $\quad \ln_0 z = \ln|z| + i\arg_0 z$, $\arg_0 z \in (0,2\pi)$




$\partial\sigma = \langle\varphi_1\rangle+\langle\varphi_2\rangle-\langle\varphi_3\rangle-\langle\varphi_4\rangle$

$\varphi_1(t) = te^{i\alpha}$ $\quad t\in\langle\varepsilon,R\rangle$

$\varphi_2(t) = Re^{it}$ $\quad t\in\langle\alpha,2\pi-\alpha\rangle$

$\varphi_3(t) = te^{i(2\pi-\alpha)}$ $\quad t\in\langle\varepsilon,R\rangle$

$\varphi_4(t) = \varepsilon e^{it}$ $\quad t\in\langle\alpha,2\pi-\varepsilon\rangle$

$\boxed{\text{Reziduová věta}}$

$\int_{\partial\sigma} \underbrace{z^{a-1}f(z)}_{F(z)} = 2\pi i\sum_{\substack{a\in\sigma\\ \text{sing. } f}}\mathrm{res}_a F(z)$

$= \int_{\langle\varphi_1\rangle}\dots + \int_{\langle\varphi_2\rangle} - \int_{\langle\varphi_3\rangle} - \int_{\langle\varphi_4\rangle}$

1) Ukážeme, kdy $\left|\int_{\langle\varphi_2\rangle-\langle\varphi_4\rangle} z^{a-1}f(z)\,dz\right| \longrightarrow 0$ pro $R\to\infty$, $\varepsilon\to0+$ $(\alpha\to0+)$

• $\left|\int_{\langle\varphi_4\rangle} z^{a-1}f(z)\,dz\right| \le \int_\alpha^{2\pi-\alpha}\varepsilon^{A-1}\varepsilon\,|f(\varepsilon e^{it})|\,e^{-Bt}\,dt \xrightarrow[\varepsilon\to0+]{A>0} 0$, f není v 0 sing.

$a=A+iB$ $\quad |z^{a-1}| = \exp\left(\mathrm{Re}[(a-1)\mathrm{Ln}z]\right) = \exp\left(\mathrm{Re}(a-1)\,\mathrm{Re}\,\mathrm{Ln}z - \mathrm{Im}(a-1)\,\mathrm{Im}\,\mathrm{Ln}z\right)$

$= \exp\left((A-1)\ln|z| - B\,\mathrm{Arg}\,z\right) = |z|^{A-1}e^{-B\,\mathrm{Arg}\,z}$

· $\left|\int_{\langle\varphi_2\rangle} z^{a-1} f(z)\,dz\right| \le R^A M_R \int_\alpha^{2\pi-\alpha} e^{-Bt}\,dt = C R^A \underbrace{M_R}_{\le \frac{\tilde{C}}{R^L}} \longrightarrow 0$

pokud $\frac{\tilde{C}}{R^L}$ a $\boxed{L>A}$

2) $\boxed{\varphi_1(t)}$ $\int_\varepsilon^R \underbrace{(te^{i\alpha})^{a-1}}_{\exp((a-1)[\ln t + i\alpha])} \underbrace{f(te^{i\alpha})}_{\downarrow\,\alpha\to0+\ 1} dt = \int_\varepsilon^R t^{a-1} \exp \underbrace{i\alpha(a-1)}_{\alpha\to0} f(te^{i\alpha})\,dt$

$\xrightarrow[\varepsilon\to0,\ \alpha\to0+]{R\to+\infty} \boxed{\int_0^\infty t^{a-1} f(t)\,dt}$

3) $\boxed{\varphi_3(t)}$ $\int_\varepsilon^R \left(te^{i(2\pi-\alpha)}\right)^{a-1} f\left(te^{i(2\pi-\alpha)}\right) e^{i(2\pi-\alpha)}\,dt$

$\left(te^{i(2\pi-\alpha)}\right)^{a-1} = \exp\left((a-1)[\ln t + i(2\pi-\alpha)]\right)$, $\quad e^{i(2\pi-\alpha)} \to e^{i2\pi} = 1$

$= \int_\varepsilon^R t^{a-1} \underbrace{e^{i(2\pi-\alpha)a}}_{\downarrow\ e^{i2\pi a}} \underbrace{f\left(te^{i(2\pi-\alpha)}\right)}_{\downarrow\,\alpha\to0+\ 1} dt$

$\xrightarrow[\varepsilon\to0+,\ \alpha\to0+]{R\to\infty} e^{i2\pi a} \int_0^\infty t^{a-1} f(t)\,dt$

$\boxed{\text{Celkově}}$

$$\left(1-e^{i2\pi a}\right) I(a) = 2\pi i \sum_{a\in\mathbb{C}} \operatorname{res}_a \left(z^{a-1} f(z)\right)$$

Předpoklady na f a na a:
- f meromorfní, nemá póly na $\mathbb{R}_0^+$
- $\operatorname{Re} a \in (0,L)$ kde L je takové, že $\max_{|z|=R} |f(z)|\, R^L$ omez. pro $R\to+\infty$

Př. Je-li f racionální a stupeň jmenovatele = stupeň čitatele + L, pak pro takové f výpočet O.K.

$I(p) = \int_0^\infty \frac{dx}{x^p(x+1)}$ $\quad p\in(0,1)$ $\quad a=(1-p)$, $a\in(0,1)$

$= \int_0^\infty x^{a-1}\frac{dx}{x+1} = \frac{2\pi i}{1-e^{2\pi a i}} \operatorname{res}_{-1} \frac{z^{a-1}}{z+1} = \frac{2\pi i}{1-e^{2\pi a i}}\, e^{ia\pi} \underbrace{e^{-i\pi}}_{=-1} = \frac{2\pi i}{e^{\pi a i}-e^{-\pi a i}} = \frac{\pi}{\sin \pi a}$ ■

$\exp\left((a-1)(\underbrace{\ln|1|}_{=0} + i\pi)\right)$

$\sin\pi a = \sin(\pi-\pi p) = \sin \pi p$

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(6) $I = \int_0^1 x^{\alpha-1}(1-x)^{-\alpha} g(x)\,dx$ ~~Má sing. v 1!~~

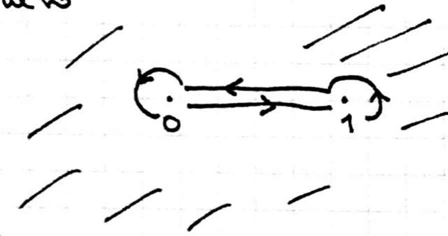
$\boxed{\text{2 metody}}$

1) $\xi = \frac{x}{1-x} \Rightarrow x = \frac{\xi}{1+\xi}$

$$I = \int_0^\infty \frac{\xi^{\alpha-1}}{(1+\xi)^{\alpha-1}} (1+\xi)^{\alpha} \frac{[1+\xi-\xi]}{(1+\xi)^2}\, g\!\left(\frac{\xi}{1+\xi}\right) d\xi$$

$$= \int_0^\infty \xi^{\alpha-1} \left(\frac{G(\xi)}{(1+\xi)}\right) d\xi$$ Mellinova transformace

2) ucho



K tomu však potřebujeme residuovou větu pro vnějšek.

~~PŘÍŠTĚ~~. POZDĚJI

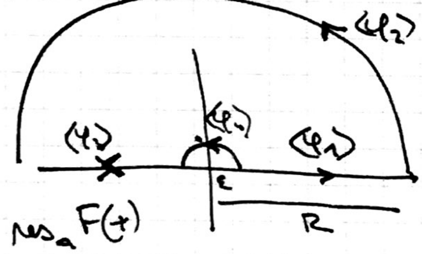
(7) Integrály typu $I = \int_0^\infty f(x)\ln x\,dx$

Požadavky na f
- f meromorfní (reálná na $\mathbb{R}$)
- f nemá pól v $\mathbb{R}$ a $M_R R\ln R \xrightarrow{R\to\infty} 0$
- f sudá

Což např. splňuje: $\Leftarrow$ f racionální sudá, st. jm $\geq$ st. č. + 2

$$\boxed{F(z) = f(z)\ln z}$$

σ:



$\partial\sigma = \langle\varphi_1\rangle + \langle\varphi_2\rangle + \langle\varphi_3\rangle - \langle\varphi_4\rangle$

Res. věta $$\int_{\langle\varphi_1\rangle} + \int_{\langle\varphi_2\rangle} + \int_{\langle\varphi_3\rangle} - \int_{\langle\varphi_4\rangle} = 2\pi i \sum_{a\in\sigma} \mathrm{res}_a F(z)$$

- $\int_{\langle\varphi_1\rangle} \dots \xrightarrow[\varepsilon\to0+]{R\to+\infty} \int_0^\infty f(x)\ln x\,dx = I$
- $\int_{\langle\varphi_3\rangle} F(z) = -\int_\varepsilon^R [\ln(t)+i\pi]\, f(-t)\,dt = -\int_\varepsilon^R f(t)\ln t\,dt - i\pi\int_\varepsilon^R f(t)\,dt$, $\quad f(-t) = f(t)$, $z = te^{i\pi}$

$$\Rightarrow\ -\int_{\langle\varphi_3\rangle} F(z)\,dz \xrightarrow[R\to\infty]{\varepsilon\to0+} I + i\pi\int_0^\infty f(t)\,dt$$

- $\left|\int_{\varphi_4} f(z)\ln z\right| = \left|\varepsilon i\int_0^\pi f(\varepsilon e^{it})[\ln\varepsilon + it]\, e^{it}\,dt\right| \leq \varepsilon\int_0^\pi |f(\varepsilon e^{it})||\ln\varepsilon + it|\,dt$

  $|f(\varepsilon e^{it})|$ ≤ omez. v okolí 0 neb v 0 nen pól.

  $\boxed{\varepsilon\ln\varepsilon \xrightarrow{\varepsilon\to0+} 0}$ $\longrightarrow 0$ pro $\varepsilon\to0+$
- $\left|\int_{\varphi_2} f(z)\ln z\right| \leq R\int_0^\pi |f(Re^{it})||\ln R + it||e^{it}|\,dt$, $\quad |e^{it}| = 1$

  $\leq \pi \max_{|z|=R}|f(z)|\, R\ln R \longrightarrow 0$ $R\to\infty$ dle předp.

L10/4

$\boxed{\text{ZÁVĚR}}$ $$2I + i\pi \int_0^\infty f(x)\,dx = 2\pi i \sum \mathrm{res}_a F(z)$$
$a \in \sigma$... horní polorovina

Tedy z porovnání reálné části dostanu $I$
z — imag. části $\int_0^\infty f(x)\,dx$

(a) $$I = \int_0^\infty \frac{\ln x}{(x^2+1)^2}$$

$$2I + i\pi \int_0^\infty \frac{dx}{(x^2+1)^2} = 2\pi i \operatorname{Res}_i \frac{\ln z}{(z^2+1)^2}$$

$$= 2\pi i \lim_{z\to i} \left[\frac{\ln z}{(z+i)^2}\right]'$$

$$= 2\pi i \lim_{z\to i} \left[\frac{1}{z}\frac{1}{(z+i)^2} + \frac{2\ln z}{(z+i)^3}\right]$$

$$= 2\pi i \left(-\frac{1}{i}\cdot\frac{1}{4} + \frac{2\cancel{i}\,\frac{\pi}{2}}{8\cancel{i}}\right)$$

$$= \frac{\pi}{2}\left(-1 + \frac{\pi}{2}i\right)$$

$$\underline{\underline{I = -\frac{\pi}{4}}} \qquad \int_0^\infty \frac{dx}{(x^2+1)^2} = \underline{\underline{\frac{\pi}{4}}}$$

∎

$$\int_{-\infty}^{+\infty} \frac{\sin ax}{\sinh x}\,dx = \operatorname{Im} \int_{-\infty}^{+\infty} \frac{e^{iax}}{\sinh x}$$

$\sinh iz = i \sin z$ a $\sin z$ má kořeny pouze $k\pi$, $k \in \mathbb{Z}$

$\sinh$ má kořeny v $ik\pi$