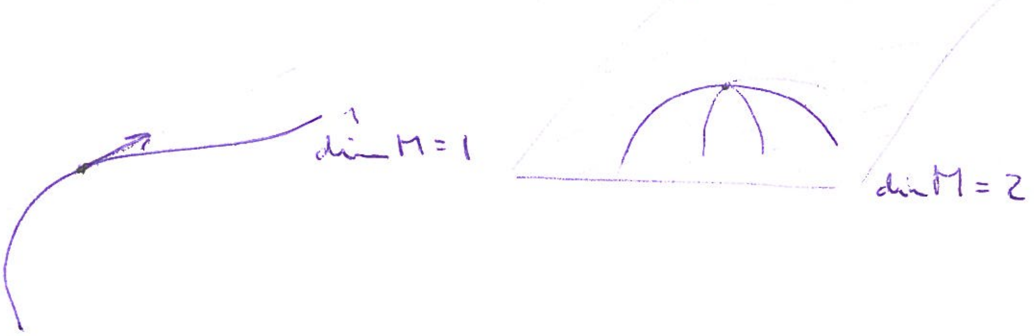
2.5 Plošný integrál 1. druhu – integrál funkce

<u>Def.</u> $M \subset \mathbb{R}^n$ q-dim. reg. plocha ($\varphi : I \to \mathbb{R}^n$ její hladká parametrizace)
Buď $x \in M$ a $\underset{u \in I^0}{\varphi(u)} = x$. Pak <u>tečný prostor T_xM k M v bodě x</u> $\equiv$ lineární obal vektorů

(*) $$\frac{\partial \varphi}{\partial u_1}(u), \dots, \frac{\partial \varphi}{\partial u_q}(u) \in \mathbb{R}^n$$



Předpoklad regularity M zaručuje, že (*) tvoří bázi T_xM

V 12.9 Nezávislost T_xM na parametrizaci

Buď M reg. plocha, $\dim M = q$. Buď $\phi, \phi' = \phi \circ \alpha$ 2 parametrizace, $\alpha : I' \to I$ difeomorfismus. $u_i = \alpha_i(u_1', \dots, u_q')$
Pro $\forall x \in M$ ($x = \phi(u)$, $x = \phi'(u')$) určují ϕ, ϕ' 2 různé báze tečného prostoru T_xM jejichž matice přechodu je Jacobiho matice α

(Δ) $$\frac{\partial \phi'}{\partial u_i'}(u') = \sum \frac{\partial(\phi \circ \alpha)}{\partial u_j} \frac{\partial \alpha_j}{\partial u_i'}(u')$$

T_xM nezávisí na volbě ϕ plochy M.

Dk (Δ) plyne z věty o derivování slož. zobrazení.
II. tvrz. plyne z (Δ), neboť báze jsou ekvivalentní. (generují stejný prostor)

Potřebujeme skalární součin (ne jej zobecnit, což se hodí pro model časoprostoru)

Def. Skalární součin se signaturou (p,q) v $\mathbb{R}^n$, $n=p+q$, definujeme pro $x=(x_1,\dots,x_n)$ a $y=(y_1,\dots,y_n)$

$$\langle x,y\rangle = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i$$

Euklidovský skalární součin: $(n,0)$

Minkowského —||— $(1,3)$ pro $n=4$

Víme: $v_1,\dots,v_n$ báze $\Rightarrow$ $\det V$ … objem rovnoběžnostěnu o stranách $\vec{v}_1,\dots,\vec{v}_n$

$$\det\left(\frac{\partial \varphi_i}{\partial x_j}\right)_{i,j=1}^{n} \qquad V = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \\ v_{n1} & \dots & v_{nn} \end{pmatrix}$$

Grammova matice $G = \left(\langle \vec{v}_i,\vec{v}_j\rangle\right)_{i,j=1}^{n} \Rightarrow \boxed{\sqrt{|\det G|} = |\det V|}$ Proč?

Jak se "vrátí" změna objemu krychličky

v_{11}

$$\langle \vec{v}_i,\vec{v}_j\rangle = \sum v_{i1}v_{j1} + v_{i2}v_{j2} + v_2$$

$$V = v_{11}v_{22} - v_{12}v_{21}$$

$$G = (v_{11}^2+v_{12}^2)(v_{22}^2+v_{21}^2) - (v_{21}v_{11} + v_{22}v_{12})^2 \quad v_{11}v_{21} - v_{12}v_{22}$$

$$v_{11}^2v_{22}^2 + v_{12}^2v_{21}^2 - 2v_{11}v_{22}v_{12}v_{21}$$

Definice (Forma objemu) Buď na $\mathbb{R}^n$ def. sk. součin se signaturou (p,q). Buď M reg. plocha dim k v $\mathbb{R}_n$, a Φ její parametrizace. Pak "formu objemu" ω se definuje

$$\omega = \sqrt{|g|}\, dx_1 \wedge \dots \wedge dx_k, \text{ kde } g = \det(g_{ij})$$

$$g = \det G \quad , \quad G_{ij} = \left\langle \frac{\partial \phi}{\partial u_i}, \frac{\partial \phi}{\partial u_j} \right\rangle_{i,j=1}^{k}$$

<u>Definice</u> <u>Integrál 1. druhu</u>

Buď $\mathbb{R}^n$, $\langle \cdot , \cdot \rangle$ se signaturou (p,q) \
Buď M reg. plocha dimenze k. } Pak $\forall f \in E^0(\Omega)$

$$\int_M f \, dS \overset{\text{df.}}{=} \int_{I \subset \mathbb{R}^k} (f \circ \phi) \sqrt{|g|} \, du_1 \dots du_k$$

Je-li M zobecněná plocha s rozkladem $M = M_1 \cup \dots \cup M_s \cup N$ pak

$$\int_M f \, dS = \sum_{i=1}^{s} \int_{M_i} f \, dS$$

<u>Věta 10</u> $\int_M f \, dS$ nezávisí na volbě M (ani na volbě M_i)

Dk ① <u>M je reg. plocha</u>, ϕ, ϕ' ... 2 parametrizace, $\alpha = \phi^{-1} \circ (\phi')$ je dif. $I' \to I$

Pak
$$g'_{ij} = \left\langle \frac{\partial \phi'}{\partial u'_i}, \frac{\partial \phi'}{\partial u'_j} \right\rangle = \sum_{k,l} \frac{\partial \alpha_k}{\partial u'_i} \frac{\partial \alpha_l}{\partial u'_j} \left\langle \frac{\partial \phi}{\partial u_k}, \frac{\partial \phi}{\partial u_l} \right\rangle$$
$$= \sum_{k,l} g_{kl} \frac{\partial \alpha_k}{\partial u'_i} \frac{\partial \alpha_l}{\partial u'_j} \Rightarrow G' = J_\alpha^T G J_\alpha$$

neboli

$$\sqrt{|g'|} = |\det J_\alpha| \sqrt{|g|} \quad (*)$$

Pak vidíme

$$\int_M f \, dS = \int_I (f \circ \phi) \sqrt{|g|} \, du_1 \dots du_k \underset{\text{věta o substituci}}{=} \int_{I'} (f \circ \phi \circ \underbrace{\phi^{-1} \circ \phi'}_{\alpha}) \sqrt{|g|} \, |\det J_\alpha| \, du'_1 \dots du'_k$$

$$\underset{(*)}{=} \int_{I'} (f \circ \phi') \sqrt{|g'|} \, du'_1 \dots du'_k = \int_M f \, dS$$

② <u>M zob. reg. plocha</u> jako u dif. forem.

<u>Pozor!</u> Integrál nezávisí na ~~o~~ zvolené orientaci

Euklidovské

Pi (1) $k=1$ $\quad \varphi: \langle a,b\rangle \to \mathbb{R}^m \Rightarrow \sqrt{|g|} =$

$v_1 = (\varphi_1', \dots, \varphi_m')$

$\langle v_1, v_1\rangle = \sum_{i=1}^{m} (\varphi_i')^2$

$$\int f\,dS \underset{\text{dle def.}}{=} \int_a^b (f\circ\varphi)(t)\sqrt{\sum_{i=1}^{m}(\varphi_i')^2(t)}\,dt$$

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(2) $k=2$ $\quad \varphi: \underset{u}{\langle a_1,b_1\rangle} \times \underset{v}{\langle a_2,b_2\rangle} \to \mathbb{R}^m$

$$G = \begin{pmatrix} \langle \vec{\varphi}_u, \vec{\varphi}_u\rangle & \langle \vec{\varphi}_u, \vec{\varphi}_v\rangle \\ \langle \vec{\varphi}_v, \vec{\varphi}_u\rangle & \langle \vec{\varphi}_v, \vec{\varphi}_v\rangle \end{pmatrix} \Rightarrow$$

$\det G = EF - G^2$, kde

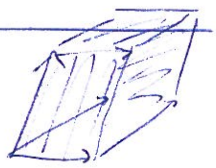
$E = \langle \vec{\varphi}_u, \vec{\varphi}_u\rangle$

$F = \langle \vec{\varphi}_v, \vec{\varphi}_v\rangle$

$G = \langle \vec{\varphi}_u, \vec{\varphi}_v\rangle$

$$\int f\,dS = \int (f\circ\varphi)(u,v)\sqrt{EF - G^2}\,du\,dv$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ $\quad \det\begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix} =$ obj. rov.



$$\begin{pmatrix} \vec{v}_1\cdot\vec{v}_1 & \vec{v}_1\cdot\vec{v}_2 & \vec{v}_1\cdot\vec{v}_3 \\ \vec{v}_2\cdot\vec{v}_1 & \vec{v}_2\cdot\vec{v}_2 & \vec{v}_2\cdot\vec{v}_3 \\ \vec{v}_3\cdot\vec{v}_1 & \vec{v}_3\cdot\vec{v}_2 & \vec{v}_3\cdot\vec{v}_3 \end{pmatrix} \overset{\text{ortog.}}{=} \begin{pmatrix} \vec{v}_1\cdot\vec{v}_1 & & \\ & \vec{v}_2\cdot\vec{v}_2 & \\ & & \vec{v}_3\cdot\vec{v}_3 \end{pmatrix} = (v_{11}^2 + v_{12}^2 + v_{13}^2)(\quad)(\quad)$$

HODGEŮV OPERÁTOR

$\Lambda^*(\mathbb{R}^n)$ ... vnější algebra $e_\emptyset, e_1, e_2, e_1\wedge e_2$

	$e_\emptyset$	e_1	e_2	e_{12}
$e_\emptyset$	$e_\emptyset$	e_1	e_2	e_{12}
e_1	e_1	0	e_{12}	0
e_2	e_2	$-e_{12}$	0	0
e_{12}	e_{12}	0	0	0

$\Lambda^*(\mathbb{R}^n) = \Lambda^0(\mathbb{R}^n) \oplus \Lambda^1(\mathbb{R}^n) \oplus \dots \oplus \Lambda^n(\mathbb{R}^n)$

$e_\emptyset \in \Lambda^0 \leftrightarrow e_1\wedge\dots\wedge e_n \in \Lambda^n$

$e_1, e_2, \dots, e_n \leftrightarrow \hat{e}_1, \hat{e}_2, \dots, \hat{e}_n$

$\hat{e}_1 \leftrightarrow e_2\wedge\dots\wedge e_n$, $\hat{e}_2 \leftrightarrow e_1\wedge e_3\wedge\dots\wedge e_n$, $\hat{e}_n \leftrightarrow e$

$e_1\wedge e_2, \dots \leftrightarrow e_3\wedge\dots\wedge e_n$

Vidíme symetrii mezi $\Lambda^k(\mathbb{R}^n)$ a $\Lambda^{n-k}(\mathbb{R}^n)$, $k = 0, 1, \dots, n$

Tuto —"— zachycuje tzv. Hodgeův operátor $*$.

Ten je jednoznačně určený pokud $\mathbb{R}^n$ má ještě další strukturu: zobecněný skalární součin signatury (p,q) • orientaci $\mathbb{R}^n$.

<u>Def.</u> Nechť na $\mathbb{R}^n$ je dán kanonický skalární součin signatury (p,q):

$$\langle x, y\rangle = \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \; y = (y_1, \dots, y_n)$$

$e_1, \dots, e_n$ .. kanonická báze v $\mathbb{R}^n$.

$\boxed{\text{Definice skalárního součinu na } \Lambda^*(\mathbb{R}^n)}$

vytvoříme báze → e_I, $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ ... ortogonální množ.

délka (velikost) e_I = $\langle e_I, e_I\rangle \overset{df.}{=} \langle e_{i_1}, e_{i_1}\rangle \cdots \langle e_{i_k}, e_{i_k}\rangle$

$\omega = \sum_I \omega_I e_I$, $\tau = \sum \tau_J e_J$; $\omega_I, \tau_J \in \mathbb{R}$, pak

$$\langle \omega, \tau\rangle = \sum_{I,J} \omega_I \tau_J \langle e_I, e_J\rangle = \sum \omega_I \tau_I \langle e_I, e_I\rangle$$

(tzv. Hodgeův operátor)

Věta 11.11. Buď $\mathbb{R}^n$ s danými kanonickým skalárním součinem signatury (p,q) a její tzv. Lebesgueova báze $e_1,\dots,e_n$. (tím i orientace).
Pak $\exists!$ lin. operátor $* : \Lambda^*(\mathbb{R}^n) \to \Lambda^*(\mathbb{R}^n)$ takový, že

1. $* : \Lambda^k(\mathbb{R}^n) \xrightarrow{\text{na}} \Lambda^{n-k}(\mathbb{R}^n)$
2. $\forall k \in \{0,\dots,n\}$ a $\forall \omega, \eta \in \Lambda^k(\mathbb{R}^n)$

$$\underset{k\text{-forma}}{\omega} \wedge [\underset{(n-k)\text{-forma}}{*(\eta)}] = \langle \omega, \eta \rangle e_1 \wedge \dots \wedge e_n$$

$\boxed{* \ \dots \ \text{Hodgeův operátor}}$

Př. M ... Minkowského prostor s $(1,3)$ - skal. součin., tzn.

$$1 = \langle e_0, e_0 \rangle = -\langle e_1, e_1 \rangle = -\langle e_2, e_2 \rangle = -\langle e_3, e_3 \rangle$$

Uvaž. vektorový prostor $T^*(M)$ s bází $e_0, e_1, e_2, e_3 = dx_0, dx_1, dx_2, dx_3$ ($= d_0, d_1, d_2, d_3$)
Uvaž. vnější algebru $\Lambda^*(T^*(M))$
Spočítejme obrazy $* : \Lambda^k(T^*(M)) \to \Lambda^{n-k}(T^*(M))$

$k = 0$: $*(e_\emptyset) = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 = d_{0123}$

$k = 1$:
$*(d_0) = d_{123}$
$*(d_1) = d_{023}$
$*(d_2) = d_{031}$
$*(d_3) = d_{012}$

$k = 2$:
$*(d_{01}) = -d_{23}$
$*(d_{02}) = -d_{31}$
$*(d_{03}) = -d_{12}$
$*(d_{12}) = d_{03}$
$*(d_{31}) = d_{02}$
$*(d_{23}) = d_{01}$

d_{10}
d_{20}
d_{30}

$k = 3$:
$*(d_{012}) = d_3$
$*(d_{023}) = d_1$
$*(d_{031}) = d_2$
$*(d_{123}) = d_0$

$k = 4$: $*(d_{0123}) = -e_\emptyset$.

$$d_{01} \wedge (*(d_{01})) = d_{01} \wedge (-d_{23}) = -d_{0123}$$

$d_0 \wedge d_1$

Mějí jsme deRhamův dif.

$$d : E^{q-1}(\Omega) \longrightarrow E^{q}(\Omega)$$

Lze definovat operátor

$$\delta : E^{q}(\Omega) \longrightarrow E^{q-1}(\Omega)$$

tak, že

$$\delta(\omega) = * d * (\omega) = \underbrace{*\underbrace{\left(d\underbrace{\left(*\underbrace{(\omega)}_{E^{q}(\Omega)}\right)}_{E^{m-q}(\Omega)}\right)}_{E^{n-(q-1)}}}_{E^{q-1}}$$

$\boxed{\delta \text{ ... tzv. kodiferenciál}}$

Příklad: Uvažme na M formu stupně 2

$$F = E_1\, dx_1 \wedge dx_0 + E_2\, dx_2 \wedge dx_0 + E_3\, dx_3 \wedge dx_0 + H_1\, dx_2 \wedge dx_3 + H_2\, dx_3 \wedge dx_1 + H_3\, dx_1 \wedge dx_2$$

$$\vec{E} = (E_1, E_2, E_3), \quad \vec{H} = (H_1, H_2, H_3)$$

Q: co vyjadřují rovnice

(1) $dF = 0$?

(2) $\delta F = 0$

Ad (1)

$$0 = \frac{\partial E_1}{\partial x_2} dx_2 \wedge dx_1 \wedge dx_0 + \frac{\partial E_1}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_0 + \frac{\partial E_2}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_0 + \frac{\partial E_2}{\partial x_3} dx_3 \wedge dx_2 \wedge dx_0$$

$$+ \frac{\partial E_3}{\partial x_1} dx_1 \wedge dx_3 \wedge dx_0 + \frac{\partial E_3}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_0 + \frac{\partial H_1}{\partial t} dx_0 \wedge dx_2 \wedge dx_3 + \frac{\partial H_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3$$

$$+ \frac{\partial H_2}{\partial t} dx_0 \wedge dx_3 \wedge dx_1 + \frac{\partial H_2}{\partial x_2} dx_2 \wedge dx_3 \wedge dx_1 + \frac{\partial H_3}{\partial t} dx_0 \wedge dx_1 \wedge dx_2 + \frac{\partial H_3}{\partial x_3} dx_3 \wedge dx_1 \wedge dx_2$$

$$\Longleftrightarrow \boxed{\begin{aligned} \frac{\partial \vec{H}}{\partial t} + \operatorname{curl} \vec{E} &= 0 \\ \operatorname{div} \vec{H} &= 0 \end{aligned}}$$

Ad 2

$$*F =$$

$$= *\left(E_1 d_{10} + E_2 d_{20} + E_3 d_{30} + H_1 d_{23} + H_2 d_{31} + H_3 d_{12}\right)$$

$$= E_1 d_{23} + E_2 d_{31} + E_3 d_{12} + H_1 d_{01} + H_2 d_{02} + H_3 d_{03}$$

$\Rightarrow$

$$d(*F) =$$

$$= \frac{\partial E_1}{\partial t} d_{023} + \frac{\partial E_1}{\partial x_1} d_{123} + \frac{\partial E_2}{\partial t} d_{031} + \frac{\partial E_2}{\partial x_2} d_{231} + \frac{\partial E_3}{\partial t} d_{012} + \frac{\partial E_3}{\partial x_3} d_{312}$$

$$+ \frac{\partial H_1}{\partial x_2} d_{201} + \frac{\partial H_1}{\partial x_3} d_{301} + \frac{\partial H_2}{\partial x_1} d_{102} + \frac{\partial H_2}{\partial x_3} d_{302} + \frac{\partial H_3}{\partial x_1} d_{103} + \frac{\partial H_3}{\partial x_2} d_{203}$$

$$= \frac{\partial E_1}{\partial t} d_{023} + \frac{\partial E_1}{\partial x_1} d_{123} + \frac{\partial E_2}{\partial t} d_{031} + \frac{\partial E_2}{\partial x_2} d_{123} + \frac{\partial E_3}{\partial t} d_{012} + \frac{\partial E_3}{\partial x_3} d_{123}$$

$$+ \frac{\partial H_1}{\partial x_2} d_{012} - \frac{\partial H_2}{\partial x_1} d_{012} + \left(\frac{\partial H_3}{\partial x_1} - \frac{\partial H_1}{\partial x_3}\right) d_{031} + \left(\frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_2}\right) d_{023}$$

$$*(d(*F)) = \frac{\partial E_1}{\partial t} d_1 + \frac{\partial E_1}{\partial x_1} d_0 + \frac{\partial E_2}{\partial t} d_2 + \frac{\partial E_2}{\partial x_2} d_0 + \frac{\partial E_3}{\partial t} d_3 + \frac{\partial E_2}{\partial x_3} d_0$$

$$\left(\frac{\partial H_1}{\partial x_2} - \frac{\partial H_2}{\partial x_1}\right) d_3 + \left(\frac{\partial H_3}{\partial x_1} - \frac{\partial H_1}{\partial x_3}\right) d_2 + \left(\frac{\partial H_2}{\partial x_3} - \frac{\partial H_3}{\partial x_2}\right) d_1$$

$$\boxed{\begin{aligned} \frac{\partial \vec{E}}{\partial t} + \operatorname{curl} \vec{H} &= 0 \\ \operatorname{div} \vec{E} &= 0 \end{aligned}}$$

$\delta F = \gamma$, kde $\gamma = \rho\, dx_0 + j_1 dx_1 - j_2 dx_2 - j_3 dx_3$, kde

ρ ... hustota náboje

$\vec{j}$ ... hustota proudu, pak dostaneme

Maxwellovy rovnice s pravou stranou

Příklad 3 (Vlnové rovnice)

Spočtěte souř. tvar rovnice $(d\delta + \delta d) f = 0$, kde $f: \mathbb{R} \to \mathbb{R}$.

a) $$(\delta d) f = \delta(df) = \delta\Big(\underbrace{\frac{\partial f}{\partial t} d_0 + \frac{\partial f}{\partial x_1} d_1 + \frac{\partial f}{\partial x_2} d_2 + \frac{\partial f}{\partial x_3} d_3}_{=\omega}\Big)$$

$$= (* d *)(\omega)$$

$$= (* d)\left[\frac{\partial f}{\partial t} d_{123} + \frac{\partial f}{\partial x_1} d_{023} + \frac{\partial f}{\partial x_2} d_{031} + \frac{\partial f}{\partial x_3} d_{012}\right]$$

$$= * \left[\frac{\partial^2 f}{\partial t^2} d_{0123} - \frac{\partial^2 f}{\partial x_1^2} d_{0123} - \frac{\partial^2 f}{\partial x_2^2} d_{0123} - \frac{\partial^2 f}{\partial x_3^2} d_{0123}\right]$$

$$= \left(\frac{\partial^2 f}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}\right)(-1) = 0 \qquad \boxed{\Box f = 0}$$

b) $$d(\delta f) = d(* d *) f = d\Big(* d\, f \underbrace{dx_0 \wedge \ldots \wedge dx_3}_{d_{0123}}\Big) = 0$$

Dk Věty (i) Jednoznačnost $\quad \varepsilon = e_1 \wedge \ldots \wedge e_n \in \Lambda^n(\mathbb{R}^n)$

Buď $*$ a $☆$ 2 takové operátory (k pevné). Cíl $\quad *(\sigma) = ☆(\sigma) \ \forall \sigma \in \Lambda^k$

Z (2) vlastnosti $\quad \omega \wedge [*\sigma - ☆\sigma] = \langle \omega, \sigma - \sigma \rangle \varepsilon = 0 \quad \forall \omega \in \Lambda^k(\mathbb{R}^n)$

Stačí tedy ukázat:

$$\boxed{\begin{array}{l} \text{Je-li pro nějaké } \alpha \in \Lambda^{n-k}(\mathbb{R}^n) \\ \qquad \omega \wedge \alpha = 0 \quad \forall \omega \in \Lambda^k(\mathbb{R}^n) \\ \text{pak} \qquad \alpha \equiv 0. \end{array}} \qquad (\Box)$$

Avšak: je-li $|I| = k$ a $|J| = n-k$, pak $e_I \wedge e_J \neq 0 \iff I \cup J = \{1, \ldots, n\} \iff J = \tilde{I}$
($\tilde{I}$ je doplňková množina)

Je-li tedy $\alpha = \sum_{|J|=n-k} \alpha_J e_J$, pak pro lib. $\omega = e_I$, $|I| = k$, platí

$$0 = e_I \wedge \alpha = \sum_J \alpha_J e_I \wedge e_J = \sum_I \alpha_{\tilde{I}} \underbrace{e_I \wedge e_{\tilde{I}}}_{\neq 0} \Rightarrow \alpha_{\tilde{I}} = 0 \quad \forall I \Rightarrow \alpha = 0$$

(ii) Existence $\quad$ Buď $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ lib., $|I| = k$.

Def. $\quad * e_I = \pm e_{\tilde{I}}$, kde

$$* e_I \overset{\text{df.}}{=} \langle e_{i_1}, e_{i_1} \rangle \ldots \langle e_{i_k}, e_{i_k} \rangle \operatorname{sgn}\begin{pmatrix} I, \tilde{I} \\ I \cup \tilde{I} \end{pmatrix} e_{\tilde{I}}$$

Takto je $*$ definováno na bázi $\Lambda^*(\mathbb{R}^n)$ a stačí jej lineárně rozšířit na celou vnější algebru. Navíc splňuje obě vlastnosti:

- $\omega = e_I$, $|I| = k$, $\sigma = e_J$, $|J| = k \Rightarrow$
  - $I \neq J \Rightarrow 0 = 0$
  - $I = J \Rightarrow \quad LS = e_I \wedge (* e_I) = \langle e_{i_1}, e_{i_1} \rangle \ldots \langle e_{i_k}, e_{i_k} \rangle \left[\operatorname{sgn}\begin{pmatrix} I\ \tilde{I} \\ I \cup \tilde{I} \end{pmatrix}\right]^2 \varepsilon$
  $= \langle e_I, e_I \rangle \varepsilon$